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ABSTRACT
This paper presents a comprehensive discussion on connected morphological operators for binary images. Introducing a connectivity on the underlying space, every image induces a partition of the space in foreground and background components. A connected operator is an operator that coarsens this partition for every input image. A connected operator is called a grain operator if it has the following 'local property': the value of the output image at a given point x is exclusively determined by the zone of the partition of the input image that contains x. Every grain operator is uniquely specified by two grain criteria, one for the foreground and one for the background components. A well-known criterion is the area criterion demanding that the area of a zone is not below a given threshold. The second part of the paper is devoted to connected filters and grain filters. It is shown that alternating sequential filters resulting from grain openings and closings are strong filters and obey a strong absorption property, two properties that do not hold in the classical non-connected case.

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Keywords and Phrases: connectivity, mathematical morphology, connectivity class, connectivity opening, grain, (geodesic) reconstruction, partition, zonal graph, connected operator, opening by reconstruction, grain operator, grain criterion, area opening, stable connected operator, strong filter, grain filter, alternating sequential filter, translation invariance.

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1. Introduction

Classical morphological operators require one or more structuring elements. In practice, such operators are either local themselves or compositions of one or more local operators. Here 'local' means that the output value at a given point (pixel, in the discrete case) is determined by the input values at a small (e.g., finite) neighbourhood. One could say that classical morphological operators act on the pixel level, albeit that such actions are intrinsically parallel.

Connected morphological operators are essentially different. They do not (or rather: cannot) change values at individual pixels, but only the values at connected regions with constant grey-level, the so-called *flat zones*. Connected operators are determined by the criteria that govern the action of the operator at the flat zones. In this paper we restrict ourselves to the binary case. Here, the flat zones are the connected components (called *grains*) of the foreground and background. Even in this simple case, criteria can be quite complex; they may depend on shape characteristics of the individual zones, but also on characteristics of zones that are adjacent.

The observation that connected operators act on the flat zone level gives rise to the following, from an image processing point of view extremely important, property: connected operators can delete boundaries, they can strengthen or weaken boundaries, *but they cannot shift boundaries nor create new boundaries*. Here 'boundary' means 'boundary between zones with different grey-level'.

Connected operators cannot introduce new discontinuities and as such they are custom-made for those image analysis and computer vision applications where contour information is important, for example image segmentation. The morphological approach towards segmentation is provided by the watershed algorithm, in all its manifestations. In practical cases, this algorithm often produces a dramatic oversegmentation due to the presence of noise. To circumvent this problem, one may take recourse to the following procedure: (*i*) find markers for the relevant regions in the image, (*ii*) modify the image (or its gradient) using these markers, (*iii*) apply the watershed algorithm. This leads to a segmentation comprising one region per marker; refer to [3] for a comprehensive description. Connected operators have a great potential with respect to automatic marker extraction (i.e., step (*i*)) [6, 10, 27]. Another branch in computer vision where connected operators have proved their usefulness is motion. The reader is referred to [21, 26] for further details.

The first systematic study on connected operators is due to Serra and Salembier [30]. However, the concept of *opening by reconstruction*, one of the first connected morphological operator studied in the literature, has emerged about ten years earlier in the beginning of the eighties [19, 20]. The importance of connectivity with regard to image processing in general, though, was recognised already by Rosenfeld in the sixties [25]. The recent work by Vincent, who succeeded in finding efficient algorithms for grey-scale reconstruction [31, 32] and the area opening [33], has given an enormous impulse to the contemporary interest in connected operators.

The current paper aims to provide a systematic discussion of the theoretical aspects of connected operators. Here we limit ourselves to the binary case; in a future publication we will deal with grey-scale images. The exposition in this paper has been inspired by the works of Serra and Salembier [30, 27], as well as Crespo, Serra, and Schafer [6, 8, 9]. In fact, the major objective of this paper is to unify the concepts found there, along with some new ones, into one consistent mathematical framework. Towards that end, we have included most of the proofs, also of those results that have been stated in one of the aforementioned papers.

We give a brief overview of the contents of this paper. We first give a short description of some basic concepts and notations from mathematical morphology in Section 2. Our exposition on connected operators starts with a discussion on connectivity classes, a beautiful concept that is due to Serra [28]. This concept includes well-known connectivities (such as 4- and 8-connectivity in $\mathbb{Z}^2$) but allows many other cases, too. Given a connectivity, we define four additional concepts: the connectivity opening (Section 4) which, for a given point x, returns the connected component of a set that contains x; reconstruction (Section 5); partitions and zonal graphs (Section 6). In this paper, the zonal graph representation will be used mainly for visualizing connected operators and grain operators. However, as we intend to demonstrate in our future work, this notion is also of theoretical interest. Furthermore, the zonal graph representation may be useful with respect to the implementation of connected operators.

Every binary or grey-scale image induces a partition of the underlying space into flat zones. In Section 7 we use such partitions to give a formal definition of a connected operator, and we present some basic methods for their construction. A rather special, yet relatively important, subclass of connected operators are the grain operators; these are connected operators that are 'local' in the sense that the output at a given point depends solely on the grain (connected component) surrounding this point. Grain operators, studied in Section 8, are completely determined by two grain criteria, one for the foreground and one for the background. In Section 9 we introduce the notion of 'stability'. Essentially, stability means that two adjacent grains in a zonal graph decomposition cannot change values simultaneously. This notion turns out useful in our study of grain filters in Section 10 and connected alternating sequential filters in Section 11. In Section 12 we make some simple observations about translation invariance. The paper is concluded with some final remarks in Section 13.

2. Terminology and notation

In this section, we recall some notation and terminology that we shall use in the sequel. For a comprehensive discussion on various theoretical concepts in mathematical morphology, the reader may refer to [14].

Given a universal set E, we denote by $\mathcal{P}(E)$ the collection of subsets of E. The notation $X \in \mathcal{P}(E)$ and $X \subseteq E$ will be used interchangeably. Given $X \subseteq E$ and $h \in E$, the expression X_h denotes the translate of X along h, i.e., $X_h = \{x + h \mid x \in X\}$. Given two sets $X, Y \subseteq E$, we denote by $X \backslash Y$ the *set difference* and by $X \triangle Y$ the *symmetric difference*.

By an *operator* we shall mean a mapping $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$. The *negative* of an operator ψ is defined as

$$\psi^*(X) = [\psi(X^c)]^c . \tag{2.1}$$

Note that ψ^* can be interpreted as ψ being applied to the bacground.

An operator δ is called a *dilation* if $\delta(\bigcup_{i \in I} X_i) = \bigcup_{i \in I} \delta(X_i)$, for an arbitrary family $\{X_i \mid i \in I\} \subseteq \mathcal{P}(E)$. An operator ε is called *erosion* if $\varepsilon(\bigcap_{i \in I} X_i) = \bigcap_{i \in I} \varepsilon(X_i)$. The operator ψ is said to be:

- *increasing* if $X \subseteq Y$ implies that $\psi(X) \subseteq \psi(Y)$, $X, Y \subseteq E$
- *translation invariant* if $\psi(X_h) = [\psi(X)]_h$, $X \subseteq E$, $h \in E$
- *extensive* if $X \subseteq \psi(X)$, $X \subseteq E$
- *anti-extensive* if $\psi(X) \subseteq X$, $X \subseteq E$
- *idempotent* if $\psi^2 = \psi$.

Here $\psi^2 = \psi \circ \psi$. An operator that is increasing and idempotent is called a (*morphological*) *filter*. An *opening* (resp. *closing*) is a filter that is anti-extensive (resp. extensive). Openings are denoted by α and closings by β. An increasing operator is called an *inf-overfilter* if $\psi(\mathsf{id} \wedge \psi) = \psi$; dually, it is called a *sup-underfilter* if $\psi(\mathsf{id} \vee \psi) = \psi$. When both equalities hold, ψ is called a *strong filter*; refer to [14, 28] for a comprehensive discussion. Every strong filter is a filter, but not vice versa. Openings and closings are strong filters. The *invariance domain* of ψ is $\mathrm{Inv}(\psi) = \{X \subseteq E \mid \psi(X) = X\}$.

Given two operators ϕ and ψ, the notation '$\phi \leq \psi$' means that $\phi(X) \subseteq \psi(X)$ for every $X \in \mathcal{P}(E)$. By $\phi \wedge \psi$ and $\phi \vee \psi$ we denote the infimum and supremum, respectively, of ϕ and ψ. That is, $(\phi \wedge \psi)(X) = \phi(X) \cap \psi(X)$ and $(\phi \vee \psi)(X) = \phi(X) \cup \psi(X)$, for every $X \subseteq E$.

The *Duality Principle*, known from the theory of partially ordered sets [4] plays an important role in mathematical morphology. It means that all concepts, definitions, and propositions occur in pairs. For example, dilation and erosion are dual concepts. And also, the dual of the proposition "if ψ is an inf-overfilter, then $\mathsf{id} \wedge \psi$ is an opening" is " if ψ is a sup-underfilter, then $\mathsf{id} \vee \psi$ is a closing".

3. Connectivity class

The notion of a connected set in E is well-defined if E is a topological space. In [28], Serra generalised this concept by the introduction of a connectivity class.

3.1. Definition. Let E be an arbitrary nonempty set. A family $\mathcal{C} \subseteq \mathcal{P}(E)$ is called a *connectivity class* if it satisfies
$(C1)$ $\varnothing \in \mathcal{C}$ and $\{x\} \in \mathcal{C}$ for $x \in E$
$(C2)$ if $C_i \in \mathcal{C}$ and $\bigcap_{i \in I} C_i \neq \varnothing$, then $\bigcup_{i \in I} C_i \in \mathcal{C}$.

Alternatively, we say that $\mathcal{C}$ defines a *connectivity* on E. An element of $\mathcal{C}$ is called a *connected set*. Note that this definition is in accordance with the definition of connected subsets of a topological space: if E is a topological space then a union of topologically connected subsets with nonempty intersection is again topologically connected [12, p.108].

In [23] Ronse compares the axioms $(C1)$-$(C2)$ with another set of axioms giving a characterization of connectivity in terms of *separating pairs of sets.*

Before we come down to concrete examples, we introduce the important subclass of connectivity classes based on adjacency.

3.2. Definition. A binary relation $\sim$ on $E \times E$ is called an *adjacency relation* if it is reflexive ($x \sim x$ for every x) and symmetric ($x \sim y$ iff $y \sim x$).

3.3. Examples.
(a) On $E = \mathbb{Z}^2$, two well-known adjacency relations are 4-adjacency and 8-adjacency.
(b) On $E = \mathbb{R}^2$, the relation '$x \sim y$ if $\|x - y\| \leq 1$' defines an adjacency relation.

Given an adjacency relation on $E \times E$, we call $x_0, x_1, \ldots, x_n$ a *path* between the points x and y if $x = x_0 \sim x_1 \sim \cdots \sim x_n = y$. Define $\mathcal{C}_\sim \subseteq \mathcal{P}(E)$ as the collection of all $C \subseteq E$ such that any two points in C can be connected by a path that lies entirely in C.

3.4. Proposition. *If $\sim$ is an adjacency relation on $E \times E$, then $\mathcal{C}_\sim$ is a connectivity class.*

Proof. $(C1)$ is obvious; we give a demonstration of $(C2)$. Let C_i, $i \in I$, be a collection of connected sets that contain the point z in their intersection. Let $x, y \in \bigcup_{i \in I} C_i$, say $x \in C_{i_1}$, $y \in C_{i_2}$. Within C_{i_1} there is a path between x and z, and within C_{i_2} there is a path between z and y. Concatenation of these paths yields a path in $\bigcup_{i \in I} C_i$ between x and y. ∎

3.5. Definition. $\mathcal{C}$ is a *strong connectivity class* if there exists an adjacency relation $\sim$ on $E \times E$ such that $\mathcal{C} = \mathcal{C}_\sim$ and E is connected. We say that E possesses a *strong connectivity.*

We present some examples.

3.6. Examples.
(a) If $\mathcal{C}$ comprises the empty set and the singletons, then $\mathcal{C}$ is a connectivity class. Observe that $\mathcal{C} = \mathcal{C}_\sim$, where $\sim$ is the trivial adjacency defined by $x \sim y$ if and only if $x = y$. However, this connectivity is not strong since E is not connected.
(b) $\mathcal{C} = \mathcal{P}(E)$ is a connectivity class, and $\mathcal{C} = \mathcal{C}_\sim$, where $\sim$ is the trivial adjacency given by $x \sim y$, for every two points $x, y \in E$. This connectivity is strong.
(c) The class $\mathcal{C}$ comprising the empty set, the singletons, and the co-finite subsets of E (a set X is co-finite if its complement X^{c} is finite) is a connectivity class. There is no underlying adjacency in this case.
(d) Define $\mathcal{C}$ as the family of sets $C \subseteq E$ with $\mathrm{card}(C) \notin \{2, 3, \ldots, n\}$, where $\mathrm{card}(C)$ denotes the number of elements of C. Then $\mathcal{C}$ is a connectivity class, but it is not generated by an adjacency relation on E.

The examples in (a) and (b) are the smallest and largest connectivity class, respectively, and we shall denote them by $\mathcal{C}_{\min}$ and $\mathcal{C}_{\max}$, respectively. Recall that the latter is a strong connectivity class.

In the following example we restrict ourselves to spaces with some additional structure.

3.7. Examples.
(a) The 4- and 8-adjacency relations (cf. Example 3.3(a)) yield the strong connectivity classes $\mathcal{C}_4$ and $\mathcal{C}_8$ in $\mathcal{P}(\mathbb{Z}^2)$, respectively.
(b) Define an adjacency relation on $\mathbb{R}^2$ by: $x \sim y$ if $x = y$ or if x and y are integer points that are 8-connected. The only connected sets that contain non-integer points are the singletons. For the subcollection $\mathcal{P}(\mathbb{Z}^2)$, the connected sets are the sets in $\mathcal{C}_8$ introduced in (a).
(c) The collection $\mathcal{C} \subseteq \mathcal{P}(\mathbb{R})$ containing the empty set, the singletons, and the intervals (a, b), where $a, b \in \mathbb{Z} \cup \{-\infty, +\infty\}$ and $a < b$, is a connectivity class.
(d) Let $\mathcal{C} \subseteq \mathcal{P}(\mathbb{R}^2)$ consist of all sets whose points cannot be separated by a straight line; see Figure 3.1. It is not difficult to verify that $\mathcal{C}$ defines a connectivity.

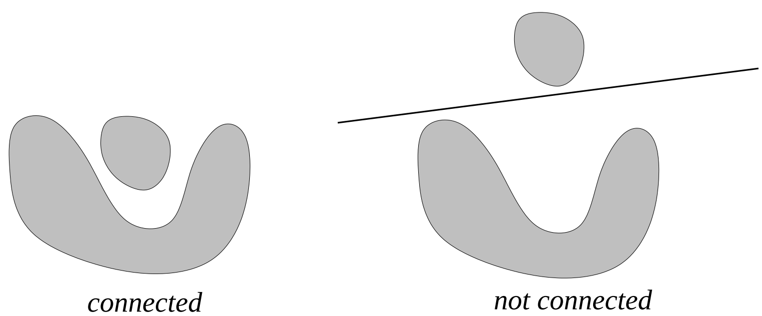


Fig. 3.1. A set is connected if its points cannot be separated by a straight line.

In fact, a set is connected iff every orthogonal projection onto a 1-dimensional space is connected in the usual sense.
(e) Say that $C \subseteq \mathbb{R}^2$ is connected if any two points in C can be joined by a polygonal line in C; see Figure 3.2. This defines a connectivity class.

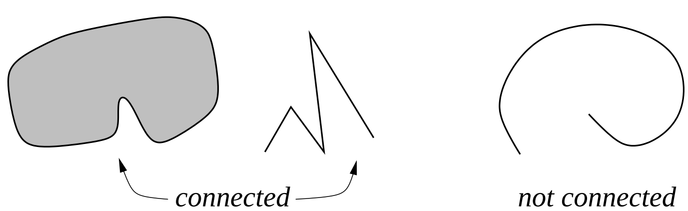


Fig. 3.2. A set C is connected if any two points in C can be joined by a polygonal line in C.

The examples (c)-(d) are adapted from [23].

There are several ways to build new connectivities from existing ones. The most important construction methods are given below in the form of propositions which are rather straightforward. For the sake of illustration, we shall prove the last one.

3.8. Proposition. *If $\mathcal{C}_k$ is a connectivity class in $\mathcal{P}(E)$ for every $k \in K$, then their intersection $\bigcap_{k\in K} \mathcal{C}_k$ is a connectivity class, too.*

3.9. Proposition. *Assume that $\mathcal{C}$ is a connectivity class in $\mathcal{P}(E)$ and let $x_0 \in E$ be fixed. The family $\mathcal{C}_0$ that consists of $\{x_0\}$ and all sets in $\mathcal{C}$ that do not contain x_0 defines a connectivity class.*

3.10. Proposition. *Assume that $\mathcal{C}$ is a connectivity class in $\mathcal{P}(E)$, let E' be a nonempty set and $\pi : E' \to E$ an arbitrary mapping. Define $\pi(X) = \{\pi(x) \mid x \in X\}$ for $X \subseteq E'$. Then $\mathcal{C}' = \{C \subseteq E' \mid \pi(C) \in \mathcal{C}\}$ is a connectivity class in $\mathcal{P}(E')$.*

3.11. Proposition. *Let E be an Abelian group and E_0 a subgroup of E. Assume that $\mathcal{C}$ is a connectivity class in $\mathcal{P}(E)$ that is invariant under translations in E_0 (i.e., $C \in \mathcal{C}$ implies that $C_x \in \mathcal{C}$ for $x \in E_0$). Let $\mathcal{C}' \subseteq \mathcal{P}(E)$ consist of the empty set, the singletons, and the sets $C \oplus E_0$, where $C \in \mathcal{C}$. Then $\mathcal{C}'$ is a connectivity class.*

3.12. Proposition. *Let $\mathcal{C}$ be a connectivity class in $\mathcal{P}(E)$ and let ψ be an increasing operator on $\mathcal{P}(E)$. Let $\mathcal{C}'$ consist of the empty set, the singletons, as well as every element $C \in \mathcal{C}$ for which $C \subseteq \psi(C)$, then $\mathcal{C}'$ is a connectivity class.*

PROOF. Let $C_i \in \mathcal{C}'$ with $\bigcap_{i\in I} C_i \neq \varnothing$, thus $\bigcup_{i\in I} C_i \in \mathcal{C}$. We have $C_i \subseteq \psi(C_i)$, hence $\bigcup_{i\in I} C_i \subseteq \bigcup_{i\in I} \psi(C_i)$. Since ψ is increasing we find that $\bigcup_{i\in I} \psi(C_i) \subseteq \psi(\bigcup_{i\in I} C_i)$. This implies that $\bigcup_{i\in I} C_i \in \mathcal{C}'$, and the result is proved. ∎

An interesting application of this last proposition is the case that $\psi = \alpha$ is an opening. For, then the condition $C \subseteq \alpha(C)$ reduces to $\alpha(C) = C$, as the inclusion $\alpha(C) \subseteq C$ trivially holds. We give an explicit example.

3.13. Example. Recall that $\mathcal{C}_8$ is the class of 8-connected subsets of $\mathbb{Z}^2$ (Example 3.7(a)). Let α be the union of the four structural openings with elementary triangles ($\{(0,0),(1,0),(0,1)\}$ and its 90°, 180°, 270° rotations). The family consisting of the empty set, the singletons, and the 8-connected sets that are open with respect to α is a connectivity class.

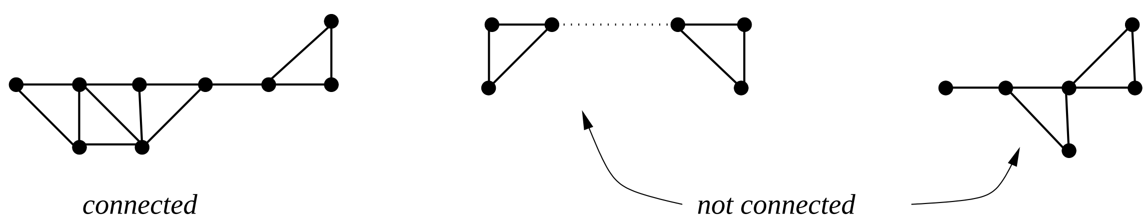


Fig. 3.3. The 8-connected subsets of $\mathbb{Z}^2$ that are invariant under the opening by the four elementary triangles constitutes a connectivity class.

In Figure 3.3 we depict three subsets of $\mathbb{Z}^2$: the first one is connected, the second one is not connected since it is not 8-connected, and the third one is not connected since $\alpha(X) \neq X$.

Another way to build new connectivities from existing ones is by means of dilation. We describe this method in detail in the next section; see Proposition 4.3.

4. Connectivity openings

By $\mathcal{C}_x$, where $x \in E$, we denote the subfamily of $\mathcal{C}$ consisting of sets C that contain the point x. Every set $X \subseteq E$ can be written as a union of connected sets that are pairwise disjoint and, moreover, this decomposition is unique. To see this, pick an element $x \in X$ and define $\gamma_x(X)$ as the union of all sets $C \in \mathcal{C}$ that contain the point x:

$$\gamma_x(X) = \bigcup \{C \in \mathcal{C} \mid x \in C \text{ and } C \subseteq X\}. \tag{4.1}$$

Since all sets C at the right hand-side contain at least one point in their intersection, namely x, their union $\gamma_x(X)$ is connected. Furthermore, we put $\gamma_x(X) = \varnothing$ if $x \notin X$. The invariance domain of γ_x comprises, besides the empty set, all connected sets that contain x. In other words,

$$\mathrm{Inv}(\gamma_x) = \mathcal{C}_x \cup \{\varnothing\}. \tag{4.2}$$

It is evident that

$$\mathcal{C} = \bigcup_{x \in E} \mathrm{Inv}(\gamma_x). \tag{4.3}$$

The following result has been established by Serra [28]; see also [14, 24].

4.1. Proposition. *Assume that $\mathcal{C}$ is a connectivity on E and let the operators γ_x on $\mathcal{P}(E)$ be defined by (4.1). The the following conditions are satisfied:*
(O1) every γ_x is an opening
(O2) $\gamma_x(\{x\}) = \{x\}$
(O3) $\gamma_x(X) \cap \gamma_y(X) = \varnothing$ or $\gamma_x(X) = \gamma_y(X)$
(O4) $x \notin X \;\Rightarrow\; \gamma_x(X) = \varnothing$

Conversely, if γ_x, $x \in E$, is a family of operators satisfying (O1) – (O4), and if $\mathcal{C}$ is defined by (4.3), then $\mathcal{C}$ defines a connectivity. Furthermore, (4.1) holds in this case.

The openings γ_x are called *connectivity openings*. Given a set $X \subseteq E$, every connected component $\gamma_x(X)$ of X is called a *grain* of X. The next results says that every connected subset of X is contained within some grain of X.

4.2. Proposition. *Given a connectivity on E and a set $X \subseteq E$. If $C \subseteq X$ is a connected set, then C is contained within some grain of X.*

Proof. Assume that Y_1, Y_2 are grains of X and that $C \cap Y_i \neq \varnothing$ for $i = 1, 2$. Then $C \cup Y_1, C \cup Y_2 \in \mathcal{C}$ and $(C \cup Y_1) \cap (C \cup Y_2) \neq \varnothing$, hence $C \cup Y_1 \cup Y_2 \in \mathcal{C}$. But this contradicts the assumption that Y_1, Y_2 are grains. ∎

An interesting method to build a new connectivity from an existing one is by means of dilation. This method was first described by Serra [28], but the formulation below are due to Ronse [24]; we refer to the latter for a proof.

4.3. Proposition. *Let $\mathcal{C}$ be a connectivity class in $\mathcal{P}(E)$ with connectivity openings γ_x. Assume that δ is an extensive dilation on $\mathcal{P}(E)$ such that $\delta(\{x\}) \in \mathcal{C}$, for every $x \in E$. Then*

$$\mathcal{C}^\delta = \{X \subseteq E \mid \delta(X) \in \mathcal{C}\} \tag{4.4}$$

is a connectivity class with $\mathcal{C} \subseteq \mathcal{C}^\delta$, and the corresponding connectivity openings are given by

$$\gamma_x^\delta = \mathsf{id} \wedge \gamma_x \delta, \quad x \in E. \tag{4.5}$$

Furthermore, the equality

$$\delta \gamma_x^\delta = \gamma_x \delta$$

holds.

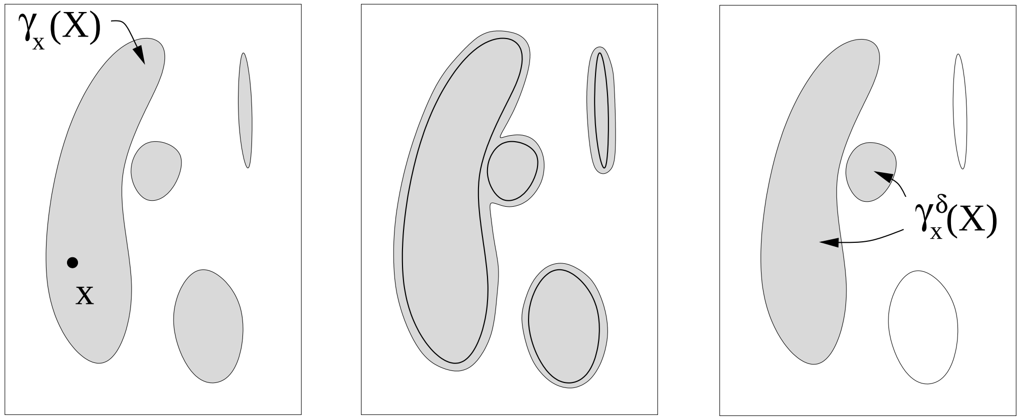


Fig. 4.1. From left to right: a set X and the grain $\gamma_x(X)$; the dilation $\delta(X)$; the grain $\gamma_x^\delta(X)$.

This proposition is illustrated in Figure 4.1.

Assume that the connectivity class $\mathcal{C}$ in Proposition 4.3 is defined by the adjacency relation $\sim$ on $E \times E$. If δ satisfies the assumptions of Proposition 4.3, one can define an adjacency relation $\overset{\delta}{\sim}$ as follows:

$$x_1 \overset{\delta}{\sim} x_2 \text{ if there exist } y_1 \in \delta(\{x_1\}),\ y_2 \in \delta(\{x_2\}) \text{ such that } y_1 \sim y_2. \tag{4.6}$$

It is rather straightforward to show that the connectivity class $\mathcal{C}^\delta$ in (4.4) is based on the adjacency $\overset{\delta}{\sim}$.

4.4. Remarks.
(a) For the family $\mathcal{C}^\delta$ in (4.4) to be a connectivity class, it is sufficient that δ is a dilation with $\delta(X) \neq \varnothing$, for every $X \subseteq E$. However, we cannot derive an explicit expression such as (4.5) for the associated connectivity class in this case.
(b) Under the assumptions of Proposition 4.3 we can show that

$$\mathcal{C}^\delta = \{X \subseteq E \mid X \subseteq C \subseteq \delta(X) \text{ for some } C \in \mathcal{C}\} \tag{4.7}$$

The inclusion '$\subseteq$' is trivial. Assume now that $X \subseteq C \subseteq \delta(X)$; we show that $\delta(X) \in \mathcal{C}$. From Proposition 4.2 we know that C lies within some grain Y of $\delta(X)$. Take $x \in X \subseteq C$. As $x \in \delta(\{x\}) \cap C$, we get that $\delta(\{x\}) \subseteq Y$, too. Therefore, $\delta(X) = \bigcup_{x \in X} \delta(\{x\}) \subseteq Y$, which yields $\delta(X) = Y$, i.e., $\delta(X) \in \mathcal{C}$.
(c) In fact, it is not difficult to show that

$$\mathcal{C}^\psi = \{X \subseteq E \mid X \subseteq C \subseteq \psi(X) \text{ for some } C \in \mathcal{C}\} \tag{4.8}$$

is a connectivity class for any increasing operator ψ, presumed that ψ is extensive on singletons, i.e., $x \in \psi(\{x\})$, for every $x \in E$.

5. Reconstruction

Given a connectivity $\mathcal{C}$ on E, we write $C \Subset X$ if C is a grain of X, i.e., $C = \gamma_x(X)$ for some $x \in X$. Note that this notation means automatically that C is connected and $C \neq \varnothing$. We define $\rho(Y \mid X)$ as the union of all grains of X that intersect Y:

$$\rho(Y \mid X) = \bigcup \{C \Subset X \mid C \cap Y \neq \varnothing\}. \tag{5.1}$$

We call $\rho(Y \mid X)$ the (*geodesic*) *reconstruction* of Y in X [14, 20]; see Figure 5.1 below for an example.

We establish the following relations between connectivity classes and reconstruction.

5.1. Proposition. *Assume that $\mathcal{C}$ is a connectivity on E with connectivity openings γ_x. The reconstruction ρ given by* (5.1) *satisfies the properties:*
$(R1)$ $Y \cap X \subseteq \rho(Y \mid X)$
$(R2)$ *$\rho(Y \mid \cdot)$ is an opening, in particular $\rho(Y \mid X) \subseteq X$*
$(R3)$ *$\rho(\cdot \mid X)$ is a dilation*
$(R4)$ *$\rho(\cdot \mid X)$ is symmetric, i.e. $y \in \rho(\{x\} \mid X) \iff x \in \rho(\{y\} \mid X)$*
$(R5)$ *$\rho(\cdot \mid X)$ is idempotent.*

There exist the following relations between γ_x and ρ:

$$\gamma_x(X) = \rho(\{x\} \mid X) \tag{5.2}$$

and

$$\rho(Y \mid X) = \bigcup_{y \in Y} \gamma_y(X) \tag{5.3}$$

Conversely, assume that $\rho(\cdot \mid \cdot) : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{P}(E)$ is such that $(R1) - (R5)$ hold. The operators γ_x on $\mathcal{P}(E)$ given by (5.2) *satisfy the properties $(O1) - (O4)$, and as such they correspond with a connectivity class $\mathcal{C}$ given by* (4.3)*. Furthermore,* (5.1) *and* (5.3) *are satisfied.*

Proof. Let $\mathcal{C}$ be a connectivity on E with connectivity openings γ_x and let ρ be given by (5.1). Then $\rho(\{y\} \mid X) = \bigcup\{C \Subset X \mid y \in C\} = \gamma_y(X)$, thus (5.2) holds. The set $\{C \Subset X \mid C \cap Y \neq \varnothing\}$ in (5.1) contains the grains of X that contain a point in Y, hence this set equals $\{\gamma_y(X) \mid y \in Y\}$. Thus (5.3) is satisfied. We demonstrate that properties $(R1) - (R5)$ hold. Property $(R1)$ is obvious. Equation (5.3) says that $\rho(Y \mid \cdot) = \bigvee_{y \in Y} \gamma_y$; since a supremum of openings is an opening [14], $(R2)$ follows. Combination of (5.2) and (5.3) gives that

$$\rho(Y \mid X) = \bigcup_{y \in Y} \rho(\{y\} \mid X),$$

which yields immediately that $\rho(\cdot \mid X)$ is a dilation. Property $(R4)$ is a straightforward consequence of (5.2) and $(O3)$. To prove $(R5)$ we use (5.3) and the fact that $\rho(\cdot \mid X)$ is a dilation (see $(R3)$):

$$\begin{aligned}
\rho(\rho(Y \mid X) \mid X) &= \rho(\bigcup_{y \in Y} \gamma_y(X) \mid X) = \bigcup_{y \in Y} \rho(\gamma_y(X) \mid X) \\
&= \bigcup_{y \in Y} \bigcup_{z \in \gamma_y(X)} \gamma_z(X) = \bigcup_{y \in Y} \bigcup_{z \in \gamma_y(X)} \gamma_y(X) \\
&= \bigcup_{y \in Y} \gamma_y(X) = \rho(Y \mid X).
\end{aligned}$$

Here we have used that $\gamma_z(X) = \gamma_y(X)$ if $z \in \gamma_y(X)$.

To prove the converse, assume that ρ satisfies $(R1) - (R5)$ and define γ_x by (5.2). First we show that $(O1) - (O4)$ hold. Property $(O1)$ follows immediately from $(R2)$. To prove $(O2)$ we must show that $\gamma_x(\{x\}) = \{x\}$. From the fact that γ_x is an opening, we get $\gamma_x(\{x\}) \subseteq \{x\}$. On the other hand, $(R1)$ yields that $x \in \gamma_x(\{x\})$. To prove $(O3)$, we first make the following observation:

$$y \in \rho(\{x\} \mid X) \;\Rightarrow\; \rho(\{x\} \mid X) = \rho(\{y\} \mid X). \tag{5.4}$$

For, (R3) implies that $\rho(\cdot \mid X)$ is increasing, hence

$$\rho(\{y\} \mid X) \subseteq \rho(\rho(\{x\} \mid X) \mid X) = \rho(\{x\} \mid X),$$

where the equality follows from $(R5)$. Furthermore, $(R4)$ yields that $x \in \rho(\{y\} \mid X)$ if $y \in \rho(\{x\} \mid X)$, and the same argument now shows that $\rho(\{x\} \mid X) \subseteq \rho(\{y\} \mid X)$, whence equality in (5.4) follows. Now, if $z \in \gamma_x(X) \cap \gamma_y(X)$, then by (5.4), $\gamma_z(X) = \gamma_x(X) = \gamma_y(X)$, which proves $(O3)$. We prove $(O4)$. Suppose $x \notin X$; we must show that $\gamma_x(X) = \varnothing$. Suppose $y \in \rho(\{x\} \mid X)$; from $(R4)$ we get that $x \in \rho(\{y\} \mid X)$. But $(R2)$ yields that $\rho(\{y\} \mid X) \subseteq X$, so $x \in X$, a contradiction. The validity of relation (5.3) is a direct consequence of $(R3)$ and definition (5.2). However, starting from the connectivity openings γ_x, relations (5.1) and (5.3) must yield the same reconstruction, and we conclude that (5.1) holds as well. This finishes the proof. ∎

Proposition 4.1 and Proposition 5.1 show that there exists three equivalent but entirely different formulations of a connectivity on E: the connectivity class $\mathcal{C}$ satisfying $(C1) - (C2)$, the connectivity openings γ_x satisfying $(O1) - (O4)$, and the reconstruction ρ satisfying $(R1) - (R5)$. Depending on the situation at hand we can work with either of them.

In practice, Y is a subset of X in the expression $\rho(Y \mid X)$. As a matter of fact, it is obvious that $\rho(Y \mid X) = \rho(Y \cap X \mid X)$, which yields the empty set if $Y \cap X = \varnothing$. The sets X and Y in $\rho(Y \mid X)$ are called the *mask* (*image*) and *marker* (*image*), respectively.

If the connectivity is based on some adjacency relation, then there exists a simple propagation algorithm for the reconstruction $\rho(Y \mid X)$:

```
R = ∅;  N = Y ∩ X;
while N ≠ ∅ do {
  R = R ∪ N;   N' = ∅;
  for x ∈ N and y ∈ X\R with y ~ x do N' = N' ∪ {y};
  N = N';
}
ρ(Y | X) = R
```

At the end of every step in the while-loop, N contains the points in X that are adjacent to points added in the previous step and that have not been found before.

The algorithm is illustrated in Figure 5.1 for the case of 8-connectivity on $\mathbb{Z}^2$.

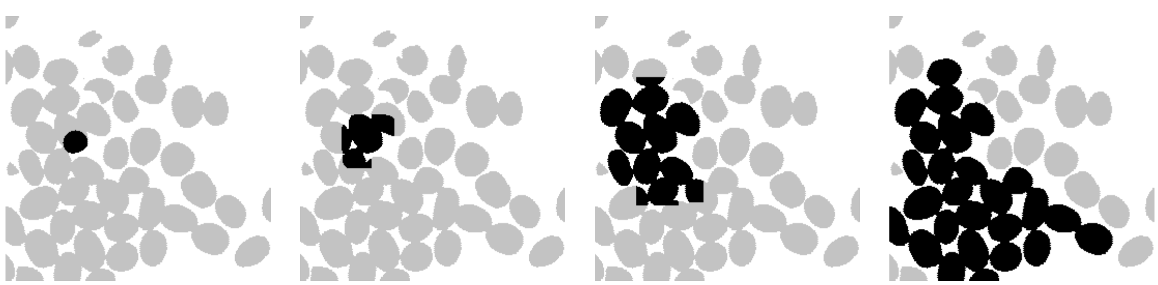


Fig. 5.1. Reconstruction algorithm for 8-connectivity. From left to right: the mask image X (grey) and the marker image Y (black); 15 iterations; 50 iterations; final result $\rho(Y \mid X)$.

From (5.2) we get that the opening γ_x can be computed with the aid of the algorithm given above; in this case we start with $N = \{x\}$.

As an illustration, we consider the case described in Proposition 4.3, where $\mathcal{C}^\delta$ is the connectivity class $\mathcal{C}^\delta = \{X \subseteq E \mid \delta(X) \in \mathcal{C}\}$. We have seen that $\gamma_x^\delta = \mathsf{id} \wedge \gamma_x \delta$ in this case. The corresponding reconstruction ρ^δ is given by

$$\rho^\delta(Y \mid X) = X \cap \rho(Y \mid \delta(X)),$$

where ρ is the reconstruction associated with connectivity $\mathcal{C}$. In fact, this relation results from a simple manipulation:

$$\begin{aligned} \rho^\delta(Y \mid X) &= \bigcup_{y \in Y} \gamma_y^\delta(X) = \bigcup_{y \in Y} (X \cap \gamma_y \delta(X)) \\ &= X \cap (\bigcup_{y \in Y} \gamma_y \delta(X)) = X \cap \rho(Y \mid \delta(X)). \end{aligned}$$

In image processing terminology, the reconstruction ρ yields a reconstruction of the foreground. Instead, one can also perform a reconstruction of the background. We call the resulting operator the *background reconstruction* or *dual reconstruction*, and denote it by ρ^* (cf. (2.1)):

$$\rho^*(Y \mid X) = \left[\rho(Y^{\mathrm{c}} \mid X^{\mathrm{c}})\right]^{\mathrm{c}}.$$

For this operator, one can derive properties dual to $(R1) - (R5)$. In particular we get that the mapping $Y \mapsto \rho^*(Y \mid X)$ is an erosion. The dual reconstruction is illustrated in Figure 5.2.

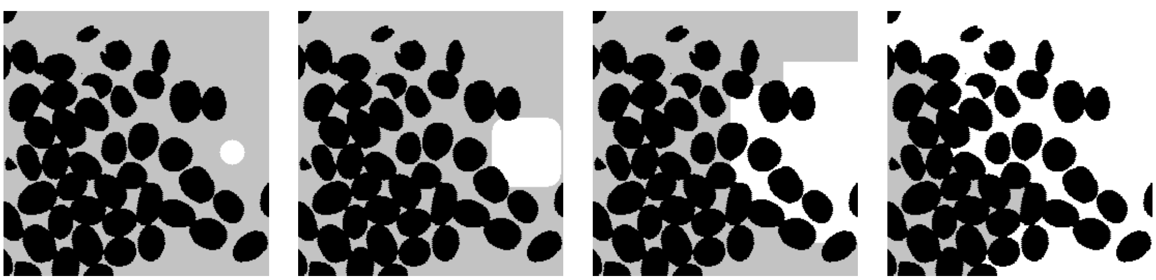


Fig. 5.2. Dual reconstruction algorithm. From left to right: the mask image X (black) and the marker image Y (grey and black); 20 iterations; 75 iterations; final result $\rho^*(Y \mid X)$ (grey and black).

Observe that

$$\rho(Y \mid X) \subseteq X \subseteq \rho^*(Y \mid X),$$

for any two sets $X, Y \subseteq E$.

6. Partitions and zonal graph representations

Having introduced the notion of a connectivity class and the derived notion of a grain of an image, we are able to give a formal definition of a connected operator. However, rather than giving this definition right away, we introduce two other concepts that, so we believe, make the definition of a connected operator easier to understand. The first concept introduced in this section is that of a partition. In words, a partition is a subdivision of the underlying space into disjoint zones.

6.1. Definition. Given a space E, a function $P : E \to \mathcal{P}(E)$ is called a *partition* of E if
(*i*) $x \in P(x), \quad x \in E$
(*ii*) $P(x) = P(y)$ or $P(x) \cap P(y) = \varnothing$, for $x, y \in E$.
We call $P(x)$ the *zone* of P that contains x.

If E is endowed with a connectivity $\mathcal{C}$ and if $P(x) \in \mathcal{C}$ for every $x \in E$, then we say that the partition P is *connected.*

Given two partitions P, P' of the space E, we say that P is *coarser* than P' (or that P' is *finer* than P) if $P'(x) \subseteq P(x)$ for every $x \in E$; see Figure 6.1 for an illustration. We denote this by $P \sqsubseteq P'$.

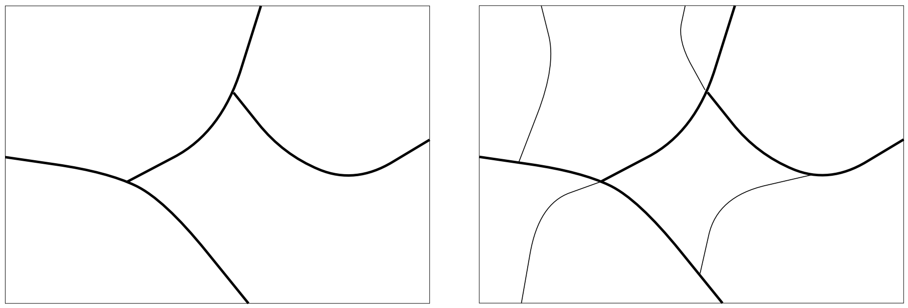


Fig. 6.1. The partition at the left is coarser than the one at the right.

The relation $\sqsubseteq$ defines a partial ordering on the set of all partitions of E. In fact, it is not difficult to show that the partially ordered set of partitions is a complete lattice [28]. The set of all connected partitions, however, does not have a lattice structure (for what follows, these observations are of no importance).

Every binary image (i.e., set) $X \subseteq E$ can be associated with a connected partition $P(X)$ where the zones of $P(X)$ are the grains of X and X^{c}. Writing $P(X, h) = P(X)(h)$, we have

$$P(X,h) = \begin{cases} \gamma_h(X), & \text{if } h \in X \\ \gamma_h(X^{\mathrm{c}}), & \text{if } h \notin X^{\mathrm{c}}. \end{cases}$$

Although this is not made explicit in our notation, the partition $P(X)$ depends upon the underlying connectivity; refer to Figure 6.3 below for an illustration.

Given a connectivity $\mathcal{C}$ on E, define a binary relation $\sim$ on $\mathcal{C} \times \mathcal{C}$ by

$$C_1 \sim C_2 \text{ if } C_1 \cup C_2 \in \mathcal{C}. \tag{6.1}$$

We say that the connected sets C_1 and C_2 are *adjacent.* By the second axiom ($C2$) of a connectivity class, we find that $C_1 \sim C_2$ if $C_1 \cap C_2 \neq \varnothing$. However, having a nonempty intersection is not a prerequisite for adjacency. The attentive reader will have noticed that we use the same notation for adjacency of connected sets as for points; see Definition 3.2. This is justified by the following observation. Let $\mathcal{C}_\sim$ be the connectivity class deriving from an adjacency $\sim$ on $E \times E$. Two sets $C_1, C_2 \in \mathcal{C}_\sim$ are adjacent in the sense of (6.1) if and only if there exist points $x_1 \in C_1, x_2 \in C_2$ such that $x_1 \sim x_2$. The latter means that $\{x_1\} \sim \{x_2\}$ in the sense of (6.1).

The zonal graph (also called *region adjacency graph* in the literature [1]) of a binary image X is a graph that takes the zones of $P(X)$ as its vertices and that uses the adjacency $\sim$ in (6.1) to define edges [22]. Furthermore, this representation specifies for each vertex whether it belongs to the foreground or the background.

6.2. Definition. Let $\mathcal{C}$ be a connectivity on E and $X \subseteq E$. The *zonal graph* of X is the triple $(P(X), \sim, I_X)$, where $I_X : P(X) \to \{0,1\}$ assigns the value 0 or 1 to every zone of $P(X)$ depending on whether this zone corresponds with a foreground or a background grain, i.e.

$$I_X(C) = \begin{cases} 1, & \text{if } C \Subset X \\ 0, & \text{if } C \Subset X^{\mathrm{c}} \end{cases}$$

An illustration of this concept is given in Figure 6.2.

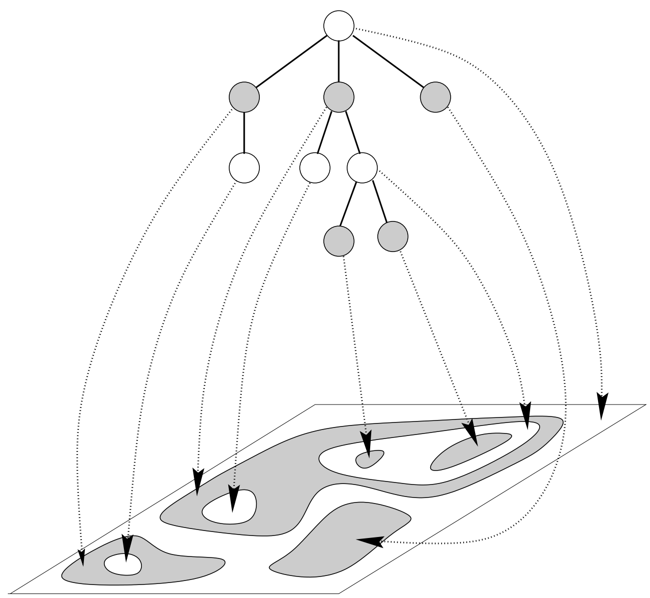


Fig. 6.2. Zonal graph (top) associated with a binary image (bottom).

Often, we refer to the value $I_X(C)$ as the colour at zone P. Note that, due to the fact that two adjacent vertices must have different colours, it suffices to specify the colour of only one vertex in each connected subgraph; see also Figure 6.3.

Different connectivities yield different zonal graphs, as is clearly illustrated by the examples in Figure 6.3. Here we consider three different connectivities on $\mathbb{Z}^2$: 4-connectivity, 8-connectivity and the so-called *chessboard connectivity.* The latter is determined by the adjacency relation: $(x,y) \sim (x',y')$ iff $|x-x'| + |y-y'| = 0$ or 2, for two points $(x,y),(x',y') \in \mathbb{Z}^2$. This means that the white fields of a chessboard are connected (as well as the black fields); however a white and a black field cannot be adjacent.

The three connectivities in Figure 6.3, although all of them are based on adjacency, are essentially different. Chessboard adjacency divides the space $\mathbb{Z}^2$ into two parts (as such, it is not a strong connectivity), and as a result also the zonal graph associated with an image X consists of two disjoint parts. For 4- and 8-adjacency the zonal graph is always connected. In fact, a much stronger result holds in the case of 8-adjacency. Recall that a *tree* is a graph without cycles [2].

6.3. Proposition. *Consider the connectivity on $\mathbb{Z}^2$ given by 8-adjacency. If $X \subseteq \mathbb{Z}^2$, then the graph $(P(X), \sim)$ is a tree.*

A proof has been given by Kong and Roscoe [18]. The example in Figure 6.3 shows that this result is not valid in the 4-adjacent case.

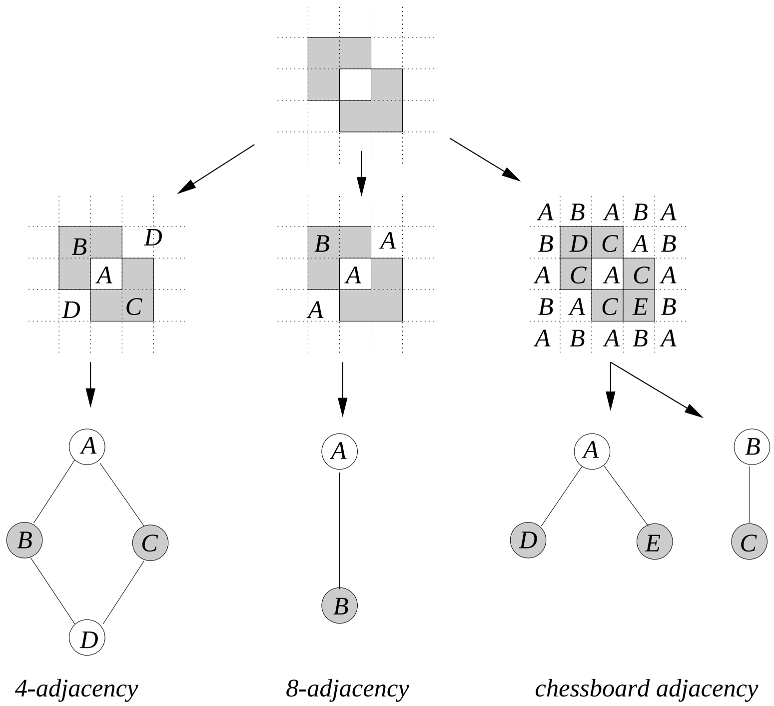


Fig. 6.3. Zonal graphs of a given set $X \subseteq \mathbb{Z}^2$ corresponding with three different connectivities.

7. Connected operators

We start with a formal definition of a connected operator.

7.1. Definition. An operator ψ on $\mathcal{P}(E)$ is *connected* if the partition $P(\psi(X))$ is coarser than $P(X)$, for every set $X \subseteq E$.

A connected operator acts on the zones of an image in an all-or-nothing way: a zone is left untouched or is changed altogether. This means in particular that boundaries of the zones can only disappear; they cannot be shifted or broken, nor can new boundaries emerge. This is nicely illustrated in Figure 7.1: here the middle image cannot be the output of a connected operator applied to the image at the left. However, the right image may result from a connected operator.
We give some simple examples.

7.2. Example.
(a) The identity operator and the complementation operator $X \mapsto X^c$ are connected (regardless of the specific connectivity).
(b) The connectivity openings γ_x are connected.
(c) For a fixed marker set Y, the reconstruction $\rho(Y \mid \cdot)$ and the dual reconstruction $\rho^*(Y \mid \cdot)$ are both connected.

We will discuss various other examples later. First we give an alternative characterisation of a connected operator [30].

7.3. Proposition. *An operator ψ is connected if and only if the symmetric difference $X \triangle \psi(X)$ consists of grains of X and X^c, for every $X \subseteq \mathbb{Z}^2$.*

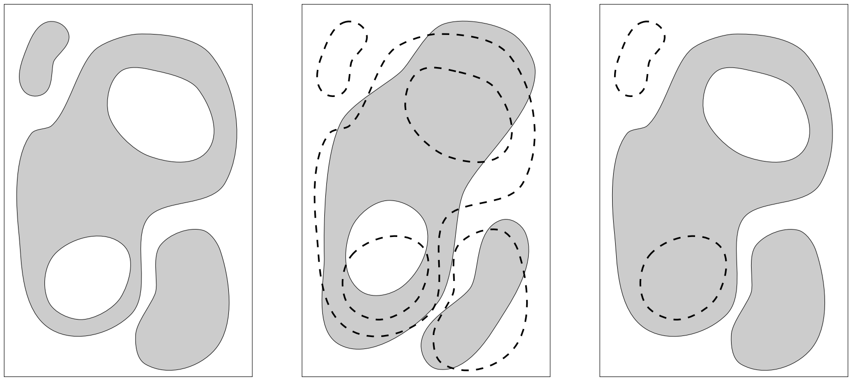


Fig. 7.1. A connected operator applied to the left image can give rise to the image at the right but not to the one in the middle.

PROOF. 'if': let $X \subseteq E$, we show that $P(X,h) \subseteq P(\psi(X),h)$, for every $h \in E$. We must distinguish between the cases $h \in X$ and $h \notin X$. We consider only the first case; the second is treated analogously. If $h \in X$, then $P(X,h) = \gamma_h(X)$. We must show that $\gamma_h(X) \subseteq P(\psi(X),h)$. Suppose $\gamma_h(X) \not\subseteq \psi(X)$; there is a point k such that $k \in \gamma_h(X)$ and $k \notin \psi(X)$. Now $k \in X\triangle\psi(X)$, which yields that $\gamma_k(X) \subseteq X\triangle\psi(X)$. However, $\gamma_k(X) = \gamma_h(X)$, whence we conclude that $\gamma_h(X) \subseteq \psi(X)^c$. Therefore $\gamma_h(X) \subseteq \gamma_h(\psi(X)^c) = P(\psi(X),h)$.

'only if': assume that ψ is connected, then $P(\psi(X))$ is coarser than $P(X)$. We must prove that for every $h \in X\triangle\psi(X)$, the entire zone $P(X,h)$ lies in $X\triangle\psi(X)$. We have to consider two cases: $h \in X$ and $h \notin X$.

$h \in X$: thus $h \notin \psi(X)$. Then $P(X,h) \subseteq P(\psi(X),h)$ leads to $\gamma_h(X) \subseteq \gamma_h(\psi(X)^c)$. But this means that $\gamma_h(X) \subseteq X\triangle\psi(X)$.

$h \notin X$: then $h \in \psi(X)$, and $P(X,h) \subseteq P(\psi(X),h)$ leads to $\gamma_h(X^c) \subseteq \gamma_h(\psi(X))$. That is, $\gamma_h(X^c) \subseteq X\triangle\psi(X)$.

∎

It is important to point out that the connectedness of a morphological operator does not only depend on the action of the operator, but also on the underlying connectivity class. This point is most clearly illustrated by considering the two extreme cases $\mathcal{C}_{\min}$ and $\mathcal{C}_{\max}$; cf. Example 3.6(a)-(b). If $\mathcal{C} = \mathcal{C}_{\min}$, then every operator on $\mathcal{P}(E)$ is connected. However, when $\mathcal{C} = \mathcal{C}_{\max}$, then the only connected operators are the identity operator $X \mapsto X$, the complementation operator $X \mapsto X^c$, and the constant operators $X \mapsto \varnothing$ and $X \mapsto E$. In those cases where it is important to indicate the particular choice of the underlying connectivity class, we will speak about $\mathcal{C}$-connected operators.

7.4. Proposition. *Consider the connectivity classes $\mathcal{C}$ and $\mathcal{C}'$, and assume that $\mathcal{C} \subseteq \mathcal{C}'$. Every $\mathcal{C}'$-connected operator is also $\mathcal{C}$-connected.*

PROOF. Given a $\mathcal{C}'$-connected operator ψ, we must show that ψ is $\mathcal{C}$-connected, that is, $X\triangle\psi(X)$ consists of $\mathcal{C}$-grains of X and X^c. Observe first that every $\mathcal{C}'$-grain of a set $Y \subseteq E$ is a union of $\mathcal{C}$-grains of this set. Since $X\triangle\psi(X)$ is a union of $\mathcal{C}'$-grains of X and X^c, it is also a union of $\mathcal{C}$-grains of X and X^c. This proves the result. ∎

In the next proposition we sum up some methods for the construction of connected operators. One of the results concerns operators resulting from substitution of given operators into a Boolean function. The idea is the following: if b is a Boolean function of n variables, and if

$\psi_1, \psi_2, \dots, \psi_n$ are operator on $\mathcal{P}(E)$, then we can define a new operator

$$\psi = b(\psi_1, \dots, \psi_n)$$

as follows:

$$\psi(X)(h) = b(\psi_1(X)(h), \dots, \psi_n(X)(h)); \tag{7.1}$$

here $X(h)$ is the indicator function associated with X, that is, $X(h)$ equals 1 if $h \in X$ and 0 otherwise. For example, if $b(u_1, \dots, u_n) = u_1 \cdot u_2 \cdot \dots \cdot u_n$, then $b(\psi_1, \dots, \psi_n) = \psi_1 \wedge \dots \wedge \psi_n$.

7.5. Proposition.
(a) *An operator ψ is connected if and only if its negative ψ^* is connected.*
(b) *If ψ_1, ψ_2 are connected, then their composition $\psi_2\psi_1$ is connected, too.*
(c) *If ψ_i is a connected operator for every i in some index set I, then the infimum $\bigwedge_{i \in I} \psi_i$ and the supremum $\bigvee_{i \in I} \psi_i$ are connected, too.*
(d) *Given a Boolean function b of n variables and n connected operators $\psi_1, \psi_2, \dots, \psi_n$, then the operator $\psi = b(\psi_1, \psi_2, \dots, \psi_n)$ is connected as well.*

PROOF. We prove (a) and (d). The other two results are proved in a similar fashion.

(a) Assume that ψ is connected, then $P(\psi(X)) \sqsubseteq P(X)$, for every $X \subseteq E$. Substituting X^c yields that

$$P(\psi(X^c)) \sqsubseteq P(X^c).$$

Using that $P(\psi^*(X)) = P(\psi(X^c)^c) = P(\psi(X^c))$, and that $P(X^c) = P(X)$, we get that

$$P(\psi^*(X)) \sqsubseteq P(X).$$

This proves the result in (a).

(d) The proof becomes obvious by the observation that the value of $\psi_i(X)(h)$ is constantly 0 or 1 on zones of the partition $P(X)$ (this value only depending on i). As a result, $\psi(X)(h)$ is constant on zones of $P(X)$, too. Therefore ψ is a connected operator. ∎

If the connectivity class is based upon adjacency, then every connected operator can be described in terms of a recolouring and merging of the corresponding zonal graph. In this paper, we confine ourselves to an informal description of this property; in a forthcoming paper it will be discussed in much greater detail. The idea is the following: since every connected operator acts on the level of the zones of the partition, it can change the value (colour) of the function I_X from 1 to 0 or vice versa. After such a recolouring, two neighbouring vertices in the zonal graph may have the same colour; such vertices can be merged into one new vertex that inherits all edges from its predecessors. This results in a new zonal graph which can then be shown to correspond to the transformed binary image. We illustrate this procedure by means of a simple example, the area operator. This operator flips the colours at zones with area less than a given threshold T ($T = 10$ in Figure 7.2).

If the connectivity class is not based upon some adjacency, this approach may fail dramatically, as the following example shows. In this example, we consider the connectivity class defined in Example 3.13. Let ψ be the connected operator that changes background grains comprising not more than one pixel. In Figure 7.3, ψ changes the value of pixel A from 0 to 1. Since this pixel corresponds to an isolated vertex in the zonal graph, it cannot be merged with the vertices B,C,D,E.

It is, on the other hand, also possible to build connected operators from a recolouring/merging procedure of zonal graphs. We illustrate the idea by means of an example. A comprehensive treatment will be postponed to a future publication.

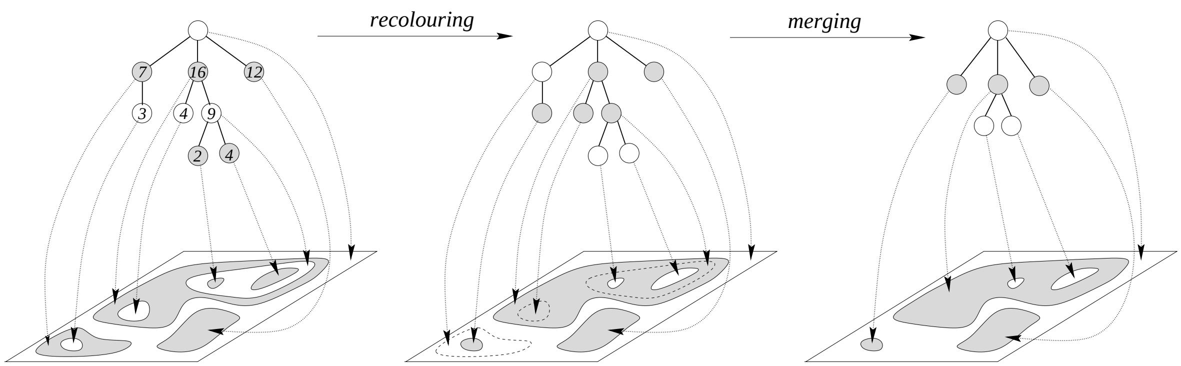


Fig. 7.2. The area operator that flips zones with area less than 10 (see the numbers printed inside the vertices at the left figure) can be interpreted as a recolouring followed by a merging of the zonal graph.

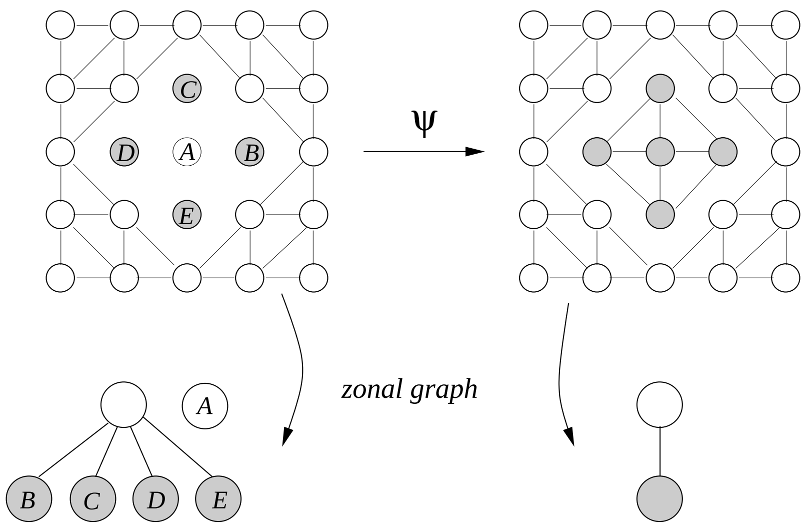


Fig. 7.3. The connected operator that changes isolated background pixels cannot be described in terms of recolouring and merging; see text.

Recall that a vertex in a tree is called a *leaf* if it possesses exactly one neighbour. For example, the tree in Figure 7.4 contains 5 leaves. We define a recolouring as follows: the colour at the leaves is flipped (from 0 to 1 and vice versa), but the colours at other vertices is left unaltered. We apply this recolouring to the zonal graph depicted at the left hand-side of Figure 7.4, and merge adjacent vertices with the same colour. The outcome is depicted at the right hand-side of Figure 7.4. The operator associated with this recolouring is connected (and self-dual).

Suppose that ϕ, ψ are operators on $\mathcal{P}(E)$, and that ψ is connected. Define the operator $\xi = \rho(\phi \mid \psi)$ by

$$\xi(X) = \rho(\phi(X) \mid \psi(X)),$$

that is, (cf.(5.3)),

$$\xi(X) = \bigcup_{h \in \phi(X)} \gamma_h(\psi(X)),$$

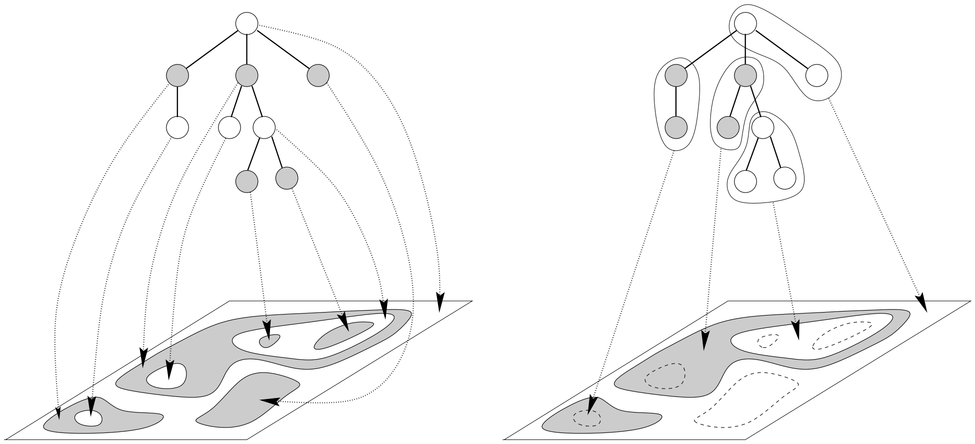


Fig. 7.4. The leaves of the tree receive the colour of their neigbour.

and in combination with the fact that $\psi(X)$ consists of grains of X and X^c, we conclude that ξ consists of grains of X and X^c; thus ξ is connected. We have the following result.

7.6. Proposition. *Assume that ϕ, ψ are operators on $\mathcal{P}(E)$ and that ψ is connected, then the operators $\xi = \rho(\phi \mid \psi)$ and $\eta = \rho^*(\phi \mid \psi)$ are connected. Furthermore,*

$$\xi \leq \psi \leq \eta.$$

Using the previous result, one can construct *connected openings* (openings that are connected operators). The basic idea is to start with an arbitrary opening and to perform a reconstruction afterwards: let α be an opening on $\mathcal{P}(E)$ and define

$$\breve{\alpha}(X) = \rho(\alpha(X) \mid X). \tag{7.2}$$

In Figure 7.5 we show an example, where $\alpha(X) = X \circ B$, B being a disk.

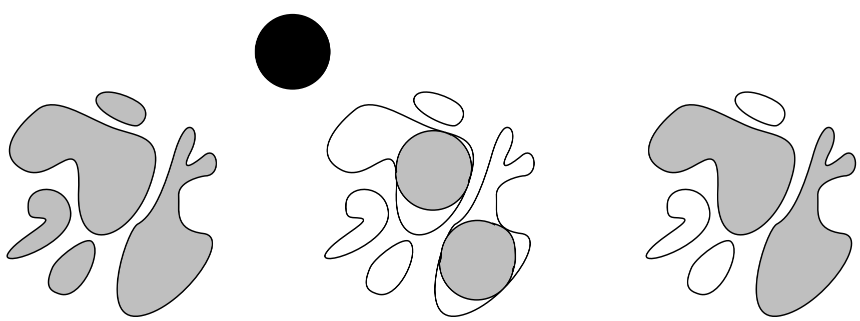


Fig. 7.5. Opening by reconstruction: the original opening is an opening by a disk (in black). From left to right: X, $\alpha(X)$, and $\breve{\alpha}(X)$.

7.7. Proposition. *If α is an opening, then $\breve{\alpha}$ is a connected opening. Moreover, α is a connected opening if and only if $\alpha = \breve{\alpha}$.*

The opening $\breve{\alpha}$ is called *opening by reconstruction*. In Proposition 8.8 we state and prove a more general version of this result.

For closings β we define

$$\check{\beta}(X) = \rho^*(\beta(X) \mid X),$$

called *closing by reconstruction*, and we can prove the dual statement of the proposition above. Note that the following duality relations hold:

$$(\alpha^*)\check{} = (\check{\alpha})^* \quad \text{and} \quad (\beta^*)\check{} = (\check{\beta})^*.$$

In the following sections we will discuss other examples of connected openings and closings.

8. Grain operators

The opening depicted in Figure 7.5 has an interesting property: it can be computed by taking the openings of the separate grains. In fact, this opening is a typical example of a class of connected operators to which we refer as *grain operators.* This class of operators has been investigated earlier by Crespo and Schafer [8] who called them *connected-component local operators.* The treatment given here is different from theirs, however.

Throughout the remainder, we use the following convention: for a statement S, the expression $[S]$ equals the Boolean value (0 or 1) indicating whether S is true or false. Thus, instead of $X(h)$ we can write $[h \in X]$.

Given a connectivity $\mathcal{C}$ on E, by a *grain criterion* we mean a mapping $u : \mathcal{C} \to \{0,1\}$. Suppose that we are given two grain criteria, u for the foreground and v for the background. Define an operator $\psi = \psi_{u,v}$ as follows:

$$\psi(X) = \bigcup\{C \mid (C \Subset X \ \text{ and } \ u(C) = 1) \ \text{ or } \ (C \Subset X^{\mathrm{c}} \ \text{ and } \ v(C) = 0)\}. \tag{8.1}$$

Thus $\psi_{u,v}$ is the operator that leaves foreground grains C for which $u(C) = 1$ and background grains C for which $v(C) = 1$ unchanged, and that flips the values at the other zones of the partition. Such operators will be called *grain operators.* The action of a grain operator is captured by Figure 8.1. By a *grain opening* we mean an opening that is at the same time a grain operator (same for closing, filter, etc).

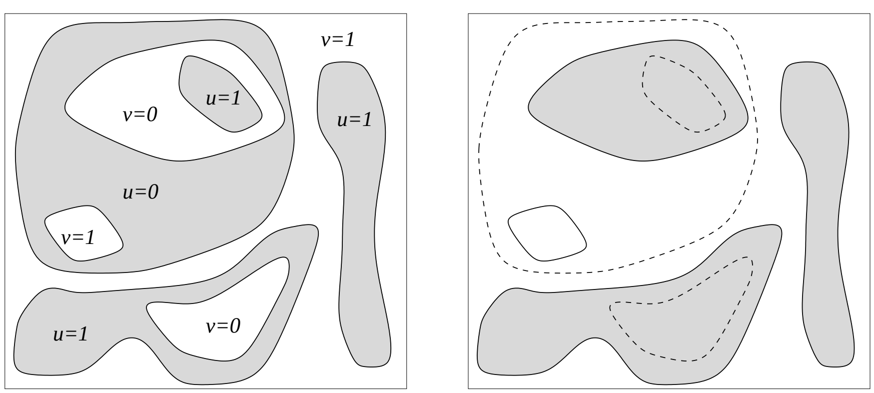


Fig. 8.1. A binary image X (left) and its transform $\psi_{u,v}(X)$ (right). In every foreground (resp. background) grain of X it is printed whether the grain criterion u (resp. v) equals 0 or 1.

Observe that u, v can be recaptured from $\psi_{u,v}$ in the following way:

$$u(C) = [C \subseteq \psi(C)] \text{ and } v(C) = [C \subseteq \psi^*(C)]. \tag{8.2}$$

Note that the second expression is equivalent to

$$v(C) = [\psi(C^c) \subseteq C^c].$$

Let us, by means of example, consider again the two extreme connectivity classes. When $\mathcal{C} = \mathcal{C}_{\max}$, then each of the four connected operators $X \mapsto X$, $X \mapsto X^c$, $X \mapsto \varnothing$, $X \mapsto E$ is a grain operator. For $\mathcal{C} = \mathcal{C}_{\min}$ the situation is less trivial. In this case, the singletons are the only non-empty connected sets. Now the foreground and background criterion can be represented by the mappings $u, v : E \to \{0, 1\}$, respectively. Let $A, B \subseteq E$ be given by $A = \{x \in E \mid u(x) = 1\}$ and $B = \{x \in E \mid v(x) = 0\}$. Then

$$\psi_{u,v}(X) = (X \cap A) \cup (X^c \cap B). \tag{8.3}$$

Thus, every grain operator is of the form (8.3), with A, B arbitrary subsets of E. Recall that, under the connectivity $\mathcal{C} = \mathcal{C}_{\min}$, every operator on $\mathcal{P}(E)$ is connected.

We write $v \equiv 1$ if $v(C) = 1$ for every $C \in \mathcal{C}$; in this case we write $\psi_{u,1}$ for $\psi_{u,v}$. Dually, $\psi_{1,v}$ represents the grain operator $\psi_{u,v}$ with $u \equiv 1$. The connectivity opening γ_h is a grain operator with $u(C) = [h \in C]$ and $v \equiv 1$. The next two examples of grain openings are more interesting.

8.1. Example (Area opening). Let $a : \mathcal{P}(E) \to \mathbb{R}_+$ be an increasing mapping, i.e., $X \subseteq Y$ implies $a(X) \le a(Y)$. Define the grain criterion $u_S(C) = [a(C) \ge S]$, where S is a given nonnegative threshold. The operator $\alpha_S = \psi_{u_S,1}$ is a grain opening. It is easy to verify by direct means that α_S is an opening, but it also follows from Proposition 8.7 given below. An important practical example is the case where a is an area measure on $\mathbb{R}^2$ or $\mathbb{Z}^2$ (in the latter case, $a(X)$ is the number of pixels of X). In this case we refer to α_S as the *area opening*. It deletes from a set X all grains with area less than S. The area opening has become very popular recently, mainly due to the efforts of Vincent [33] who invented a fast algorithm for the area opening, both for binary and grey-scale images.

8.2. Example (Opening by reconstruction). For simplicity we restrict ourselves here to 8-connectivity on $\mathbb{Z}^2$. Let $B \subseteq \mathbb{Z}^2$ be a connected structuring element and consider the grain criterion $u(C) = [C \ominus B \neq \varnothing]$. The operator $\alpha_u = \psi_{u,1}$ is a grain opening; in fact, it is the opening $\breve{\alpha}(X) = \rho(\alpha(X) \mid X)$, where α is the structural opening $\alpha(X) = X \circ B$; cf. Proposition 7.7. If B is not connected, then we only have the inequality $\alpha_u \le \breve{\alpha}$; refer to Proposition 12.4 for a precise statement.

Similarly we can build area closings and closings by reconstruction.

Given a collection of grain operators, we can build new grain operators using supremum, infimum, negation, and Boolean functions; see also Proposition 7.5. We use the following notation: U and V map a grain operator onto the corresponding foreground and background criterion, respectively; thus $U(\psi_{u,v}) = u$ and $V(\psi_{u,v}) = v$. Note that (8.2) guarantees uniqueness of u and v.

We define infima, suprema and Boolean functions of criteria in the usual way, namely pointwise. For example, if $u_1, u_2, \ldots, u_n$ are grain criteria and b a Boolean function, then $u = b(u_1, u_2, \ldots, u_n)$ is the criterion given by $u(C) = b(u_1(C), u_2(C), \ldots, u_n(C))$.

8.3. Proposition.
(*a*) *If ψ is a grain operator then ψ^* is a grain operator, too, and*

$$U(\psi^*) = V(\psi) \text{ and } V(\psi^*) = U(\psi).$$

In other words $\psi^*_{u,v} = \psi_{v,u}$. In particular, $\psi_{u,v}$ is self-dual if and only if $u = v$.
(*b*) *Assume that ψ_i, $i \in I$, are grain operators, then $\bigwedge_{i\in I} \psi_i$ is grain operator and*

$$U(\bigwedge_{i\in I} \psi_i) = \bigwedge_{i\in I} U(\psi_i) \text{ and } V(\bigwedge_{i\in I} \psi_i) = \bigvee_{i\in I} V(\psi_i).$$

Similarly, $\bigvee_{i\in I} \psi_i$ is a grain operator and

$$U(\bigvee_{i\in I} \psi_i) = \bigvee_{i\in I} U(\psi_i) \text{ and } V(\bigvee_{i\in I} \psi_i) = \bigwedge_{i\in I} V(\psi_i).$$

(*c*) *Assume that $\psi_1, \psi_2, \ldots, \psi_n$ are grain operators and that b is a Boolean function of n variables, then $\psi = b(\psi_1, \psi_2, \ldots, \psi_n)$ is a grain operator, and*

$$U(\psi) = b(U(\psi_1), \ldots, U(\psi_n)) \text{ and } V(\psi) = b^*(V(\psi_1), \ldots, V(\psi_n)).$$

Here b^* denotes the negative of b given by $b^*(u_1, \ldots, u_n) = 1 - b(1 - u_1, \ldots, 1 - u_n)$.

PROOF. We prove (c); the results in (a) and (b) are proved in a similar way. We put $u_i = U(\psi_i)$ and $v_i = V(\psi_i)$. Furthermore, $u = b(u_1, u_2, \ldots, u_n)$ and $v = b^*(v_1, v_2, \ldots, v_n)$. We must show that $\psi = \psi_{u,v}$.

Let $C \Subset X$; for every $h \in C$ we have $\psi_i(X)(h) = u_i(C)$. Recalling the expression for ψ in (7.1), we get

$$\begin{aligned}\psi(X)(h) &= b(\psi_1(X)(h), \ldots, \psi_n(X)(h)) \\ &= b(u_1(C), \ldots, u_n(C)) \\ &= u(C).\end{aligned}$$

Let $C \Subset X^c$; for every $h \in C$ we have $\psi_i(X)(h) = 1 - v_i(C)$. Hence

$$\begin{aligned}\psi(X)(h) &= b(\psi_1(X)(h), \ldots, \psi_n(X)(h)) \\ &= b(1 - v_1(C), \ldots, 1 - v_n(C)) \\ &= 1 - b^*(v_1(C), \ldots, v_n(C)) \\ &= 1 - v(C).\end{aligned}$$

These two expressions for $\psi(X)(h)$, $h \in C$, where C is a foreground resp. background grain, yield that $\psi = \psi_{u,v}$. ∎

In general, however, a composition of grain operators does not yield a grain operator. Consider the self-dual grain operator $\psi = \psi_{u,u}$ on $\mathcal{P}(\mathbb{R}^2)$, where u is the area criterion $u(C) = [a(C) \geq 20]$ and $a(C)$ is the area of C. This operator flips the value at the foreground and background grains with area less than 20. In Figure 8.2 the operator ψ^2 is applied to two different sets X and Y. If ψ^2 were a grain operator, the value of $\psi^2(X)$ and $\psi^2(Y)$ at the grain with the thick boundary ought to be the same; however, the value is 1 for $\psi^2(X)$ and 0 for $\psi^2(Y)$. Therefore ψ^2 is not a grain operator.

This example illustrates also quite nicely how grain operators act on a binary image, or better, the corresponding zonal graph. If ψ is a grain operator, then the value $\psi(X)(h)$ is completely determined by the value $X(h)$ and the vertex $P(X, h)$ of the zonal graph; information about adjacent vertices is irrelevant. This property is captured by the following proposition.

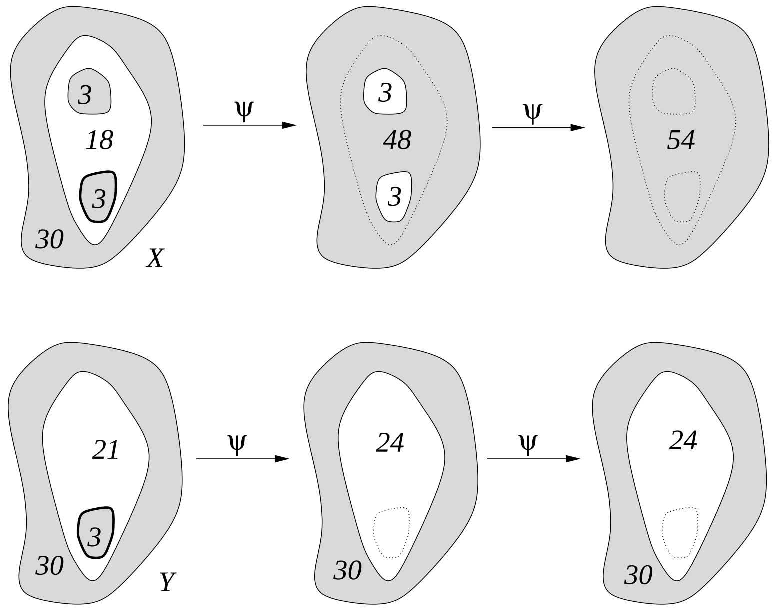


Fig. 8.2. The operator ψ that changes the colour of zones with area less than 20 is a grain operator, but ψ^2 is not.

8.4. Proposition. *A connected operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is a grain operator if and only if it has the following property: if $h \in E$ and $X, Y \subseteq E$, are such that $X(h) = Y(h)$ and $P(X,h) = P(Y,h)$, then $\psi(X)(h) = \psi(Y)(h)$.*

PROOF. 'only if': easy.

'if': Suppose that ψ is a connected operator with the given property. Define u, v by

$$u(C) = [C \subseteq \psi(C)] \text{ and } v(C) = [\psi(C^c) \subseteq C^c];$$

we must show that $\psi = \psi_{u,v}$. Let $X \subseteq E$ and $h \in E$, we demonstrate that $\psi(X)(h) = \psi_{u,v}(X)(h)$. We consider only the case that $h \in X$; the case $h \in X^c$ is treated similarly. Assume therefore that $h \in C \Subset X$. There are two possibilities:

(*i*) $u(C) = 1$: then $h \in \psi_{u,v}(X)$. Furthermore, $u(C) = 1$ means that $C \subseteq \psi(C)$. Since $X(h) = C(h) = 1$ and $P(X,h) = P(C,h) = C$, we get that $\psi(X)(h) = \psi(C)(h) = 1$, i.e., $h \in \psi(X)$.

(*ii*) $u(C) = 0$: then $h \notin \psi_{u,v}(X)$. Since $X(h) = C(h) = 1$ and $P(X,h) = P(C,h) = C$, we get that $\psi(X)(h) = \psi(C)(h) = 0$ because $C \subseteq \psi(C)^c$. Thus $h \notin \psi(X)$. ∎

In fact, in [15] we used this characterisation of grain operators as a definition and showed that every grain operator is of the form $\psi = \psi_{u,v}$, with u and v given by (8.2). Obviously, the operator depicted in Figure 7.4, where the values at the leaf of a tree are flipped, is not a grain operator. To determine whether a vertex is a leaf, one needs information about the neighbours of this vertex: "is there one or more than one neighbour?"

The next problem that we address here is the increasingness of grain operators. A criterion $u : \mathcal{C} \to \{0,1\}$ is said to be *increasing* if $u(C) \leq u(C')$ for $C, C' \in \mathcal{C}$ with $C \subseteq C'$. It is tempting to suppose that $\psi_{u,v}$ is increasing if both criteria u and v are increasing. A first counterexample to this supposition is given in Figure 8.3, where $E = \mathbb{Z}^2$ endowed with 8-connectivity.

A second counterexample is obtained by examining the connectivity class $\mathcal{C}_{\min}$, in which case every criterion is increasing. We have seen that every grain operator is of the form $\psi(X) =$

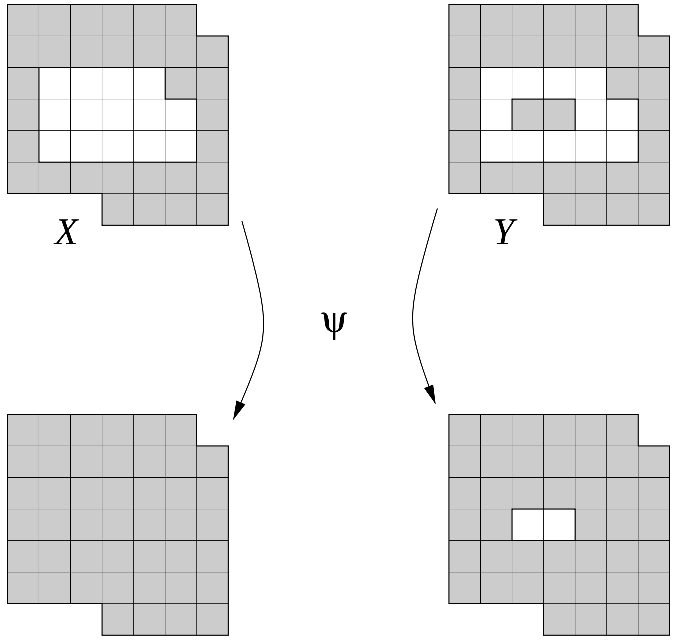


Fig. 8.3. The grain operator $\psi = \psi_{u,u}$, where $u(C) = [\mathrm{area}(C) \geq 15]$, is not increasing. Indeed, $X \subseteq Y$ but $\psi(X) \not\subseteq \psi(Y)$.

$(X \cap A) \cup (X^{\mathrm{c}} \cap B)$; here $A = \{x \in E \mid u(x) = 1\}$ and $B = \{x \in E \mid v(x) = 0\}$. This operator is increasing if and only if $B \subseteq A$, in which case ψ reduces to $\psi(X) = (X \cap A) \cup B$.

The following result shows that we need an extra condition.

8.5. Proposition. *The grain operator $\psi_{u,v}$ is increasing if and only if both u and v are increasing criteria, and the following condition holds:*

$$u(\gamma_h(X \cup \{h\})) \vee v(\gamma_h(X^{\mathrm{c}} \cup \{h\})) = 1, \tag{8.4}$$

if $X \subseteq E$ and $h \in E$.

PROOF. 'if': assume that u, v obey the conditions above; we show that $\psi = \psi_{u,v}$ is increasing. Let $X \subseteq Y$; we must show that $\psi(X) \subseteq \psi(Y)$. Take $h \in \psi(X)$. Three cases are to be distinguished:

1. $h \in X$: put $C = \gamma_h(X)$, then $C \Subset X$ and $C \subseteq C' = \gamma_h(Y)$. As $h \in \psi(X)$, we have $u(C) = 1$ and, by the increasingness of u, $u(C') = 1$ as well. This implies that $h \in \psi(Y)$.

2. $h \notin Y$: put $C' = \gamma_h(Y^{\mathrm{c}})$ and $C = \gamma_h(X^{\mathrm{c}})$, then $C' \subseteq C$ since $Y^{\mathrm{c}} \subseteq X^{\mathrm{c}}$. From the fact that $h \in \psi(X)$ we conclude that $v(C) = 0$ and thus $v(C') = 0$. We get that $h \in \psi(Y)$.

3. $h \in Y$ and $h \notin X$: suppose $h \notin \psi(Y)$, then $u(\gamma_h(Y)) = 0$. Now (8.4) implies that $v(\gamma_h(Y^{\mathrm{c}} \cup \{h\})) = 1$. Obviously, $\gamma_h(Y^{\mathrm{c}} \cup \{h\}) \subseteq \gamma_h(X^{\mathrm{c}})$, and since v is increasing, we get that $v(\gamma_h(X^{\mathrm{c}})) = 1$. However, this implies that the grain $\gamma_h(X^{\mathrm{c}})$ does not lie in $\psi(X)$, contradicting $h \in \psi(X)$. Thus we conclude that $h \in \psi(Y)$.

'only if': assume that $\psi = \psi_{u,v}$ is increasing. First we show that u is an increasing grain criterion. The proof that v is increasing is analogous. Let $C \subseteq C'$ be connected, then $\psi(C) \subseteq \psi(C')$. Suppose that $u(C) = 1$, then $C \subseteq \psi(C)$, hence $C \subseteq \psi(C')$. Thus we get that $C \subseteq C' \cap \psi(C')$, and we conclude that $u(C') = 1$ since otherwise $C' \cap \psi(C') = \varnothing$. Thus it remains to show (8.4). Let $X \subseteq E$ and $u(\gamma_h(X \cup \{h\})) = 0$; we must show that $v(\gamma_h(X^{\mathrm{c}} \cup \{h\})) = 1$.

Indeed, since $h \notin \psi(X \cup \{h\})$ and ψ is increasing, it follows that $h \notin \psi(X \backslash \{h\})$. This means that $v(P(X \backslash \{h\}, h)) = 1$. Now

$$P(X \backslash \{h\}, h) = \gamma_h((X \backslash \{h\})^c) = \gamma_h(X^c \cup \{h\}).$$

This yields the result. ∎

Indeed, for $\mathcal{C} = \mathcal{C}_{\min}$, condition (8.4) amounts to $u(h) \vee v(h) = 1$, yielding that $h \in A$ or $h \in B^c$, for every $h \in E$, i.e., $B \subseteq A$.

The area criterion of Example 8.1, $u(C) = [a(C) \geq S]$, and the structural criterion of Example 8.2, $u(C) = [C \ominus B \neq \varnothing]$ that leads to the opening by reconstruction, are both increasing. A criterion on $\mathcal{P}(\mathbb{Z}^2)$ that is not increasing is $u(C) = [\text{perimeter}(C) \geq S]$, where perimeter$(C)$ equals the number of boundary pixels in C. Also $u(C) = [\text{area}(C)/(\text{perimeter}(C))^2 \geq k]$, a criterion that provides a measure for the circularity of C, is nonincreasing. In [5] Breen and Jones discuss some other nonincreasing criteria.

We conclude this section with some results on extensive and anti-extensive grain operators, in particular, grain openings and closings. In Figure 8.2 we have presented an example showing that composition of two grain operators does not yield a grain operator in general. However, we do get some interesting results in the case where both operators are (anti-) extensive. We start with a lemma.

8.6. Lemma. *Let u_1, u_2 be two grain criteria, then*

$$\psi_{u_2,1}\psi_{u_1,1} = \psi_{u_1 \wedge u_2,1}.$$

PROOF. It is easy to establish the following relation:

$$C \Subset \psi_{u,1}(X) \text{ iff } C \Subset X \text{ and } u(C) = 1.$$

Thus

$$\begin{aligned}\psi_{u_2,1}\psi_{u_1,1}(X) &= \{C \in \mathcal{C} \mid C \Subset \psi_{u_1,1} \text{ and } u_2(C) = 1\} \\ &= \{C \in \mathcal{C} \mid C \Subset X \text{ and } u_2(C) = 1 \text{ and } u_1(C) = 1\} \\ &= \{C \in \mathcal{C} \mid C \Subset X \text{ and } (u_1 \wedge u_2)(C) = 1\}\end{aligned}$$

This yields the result. ∎

Grain operators of the form $\psi_{u,1}$ are anti-extensive, and, moreover, every anti-extensive grain operator is of this form. Combining Lemma 8.6 with Proposition 8.3(b), we arrive at the following identities:

$$\psi_{u_2,1}\psi_{u_1,1} = \psi_{u_1,1}\psi_{u_2,1} = \psi_{u_1 \wedge u_2,1} = \psi_{u_1,1} \wedge \psi_{u_2,1} \tag{8.5}$$

$$\psi_{1,v_2}\psi_{1,v_1} = \psi_{1,v_1}\psi_{1,v_2} = \psi_{1,v_1 \wedge v_2} = \psi_{1,v_1} \vee \psi_{1,v_2} \tag{8.6}$$

Taking $u_1 = u_2 = u$ in (8.5) and $v_1 = v_2 = v$ in (8.6), respectively, we get

$$\psi_{u,1}^2 = \psi_{u,1} \text{ and } \psi_{1,v}^2 = \psi_{1,v} \tag{8.7}$$

i.e., every (anti-) extensive grain operator is idempotent. Using Proposition 8.5 we arrive at the following result.

8.7. Proposition. *Let u, v be increasing grain criteria. Then $\psi_{u,1}$ is a grain opening and $\psi_{1,v}$ a grain closing.*

We write

$$\alpha_u = \psi_{u,1} \text{ and } \beta_v = \psi_{1,v}.$$

Note that we have the duality relation (cf. Proposition 8.3(a))

$$\alpha_u^* = \beta_u.$$

Specialising (8.5) to grain openings we find that

$$\alpha_{u_1}\alpha_{u_2} = \alpha_{u_2}\alpha_{u_1} = \alpha_{u_1 \wedge u_2}.$$

In particular, we have

$$\alpha_u \gamma_h = \gamma_h \alpha_u$$

for every grain operator α_u and every $h \in E$. This follows from the fact that γ_h is a grain opening with foreground criterion $u(C) = [h \in C]$. Taking the supremum over all $h \in E$ we get the identity

$$\alpha_u = \bigvee_{h \in E} \alpha_u \gamma_h,$$

which expresses that a grain opening can be evaluated grain by grain.

We conclude this section with the following generalisation of Proposition 7.7.

8.8. Proposition. *If α_u is a grain opening and α is an opening $\leq \alpha_u$, then $\breve{\alpha} = \rho(\alpha \mid \alpha_u)$ is a connected opening.*

Proof. From Proposition 7.6 we know that $\breve{\alpha}$ is connected. We must show that $\breve{\alpha}$ is an opening. It is evident that $\breve{\alpha}$ is increasing and that $\alpha(X) \subseteq \breve{\alpha}(X) \subseteq \alpha_u(X)$. Therefore $\breve{\alpha}^2 \leq \breve{\alpha}$. We must show that $\breve{\alpha}^2 \geq \breve{\alpha}$. Clearly, $\breve{\alpha}(X)$ is a union of grains of $\alpha_u(X)$, that is, grains of X that intersect with $\alpha(X)$ and satisfy criterion u:

$$\breve{\alpha}(X) = \{C \Subset X \mid C \cap \alpha(X) \neq \varnothing \text{ and } u(C) = 1\}.$$

It follows immediately that $\alpha_u \breve{\alpha}(X) = \breve{\alpha}(X)$. For $\breve{\alpha}^2(X)$ we find:

$$\begin{aligned} \breve{\alpha}^2(X) &= \rho(\alpha\breve{\alpha}(X) \mid \alpha_u\breve{\alpha}(X)) \supseteq \rho(\alpha^2(X) \mid \breve{\alpha}(X)) \\ &= \rho(\alpha(X) \mid \breve{\alpha}(X)) = \bigcup_{h \in \alpha(X)} \gamma_h(\breve{\alpha}(X)). \end{aligned}$$

Using that $\gamma_h(\breve{\alpha}(X)) = \gamma_h(\alpha_u(X))$ for $h \in \alpha(X)$, we get that $\breve{\alpha}^2(X) \supseteq \bigcup_{h \in \alpha(X)} \gamma_h(\alpha_u(X)) = \breve{\alpha}(X)$. This proves that $\breve{\alpha}$ is an opening. ∎

For completeness we point out that the analogue of Proposition 7.4 for grain operators does not hold. We leave it as an exercise to the reader to find counterexamples.

9. Stable connected operators

The stability concept for connected operators was introduced by Crespo et al [10] and studied in more detail in [8]. In these studies, however, one speaks about 'adjacency stability'. Before stating the formal definition of this concept, we give an intuitive explanation. A connected operator is stable if it cannot change two adjacent vertices (with values 0 and 1) in the zonal graph associated with some binary image.

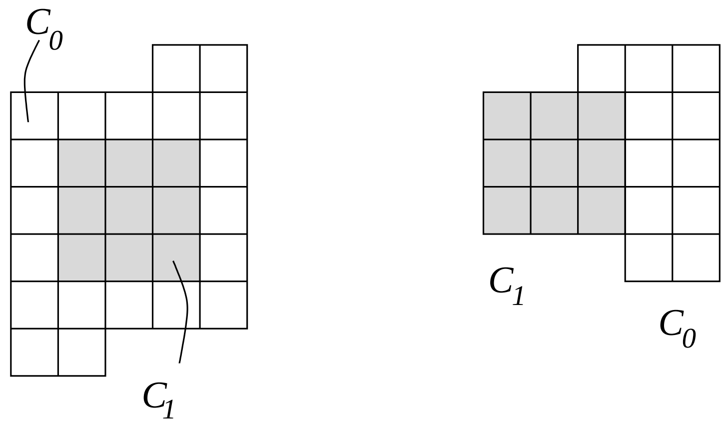


Fig. 9.1. C_1 consists of the grey pixels, C_0 of the white pixels. $C_1 \approx C_0$ at the left but not at the right.

We introduce the following notations. Let $X \subseteq E$ and $C_1, C_0 \in \mathcal{C}$; we write $C_1 \overset{X}{\approx} C_0$ if $C_1 \Subset X$, $C_0 \Subset X^{\mathrm{c}}$ and $C_1 \sim C_0$. Furthermore, $C_1 \approx C_0$ means that $C_1 \overset{X}{\approx} C_0$ for some $X \subseteq E$. Refer to Figure 9.1 for an example in the 2-dimensional discrete case.

9.1. Definition. A connected operator ψ on $\mathcal{P}(E)$ is *stable* if for every $X \subseteq E$ the following holds:

$$C_1 \overset{X}{\approx} C_0 \;\Rightarrow\; C_1 \subseteq \psi(X) \text{ or } C_0 \subseteq \psi(X)^{\mathrm{c}}. \tag{9.1}$$

Indeed, (9.1) means that ψ cannot change the colour at C_1 and C_0 simultaneously. If $\mathcal{C} = \mathcal{C}_{\min}$, then every connected operator is stable. In fact, $C_1 \overset{X}{\approx} C_0$ is never satisfied in this case so (9.1) trivially holds. However, if $\mathcal{C} = \mathcal{C}_{\max}$, the operators $X \mapsto X$, $X \mapsto E$, $X \mapsto \varnothing$ are connected and stable, whereas the operator $X \mapsto X^{\mathrm{c}}$ is connected but not stable.

The following result is evident.

9.2. Proposition. *Every (anti-) extensive connected operator is stable.*

We give a condition that is slightly stronger than the stability condition. For strong connectivity classes these two conditions are equivalent.

9.3. Proposition.

(*a*) *Every connected operator ψ satisfying*

$$\gamma_x(\mathsf{id} \vee \psi) = \gamma_x \vee \gamma_x\psi, \;\; x \in E \tag{9.2}$$

is stable.

(*b*) *On the other hand, if $\mathcal{C}$ is a strong connectivity class, then every stable operator satisfies* (9.2).

Proof. Observe first that the inequality '$\geq$' in (9.2) is trivially satisfied.

(a): Assume that (9.2) holds; we show that ψ is stable. Suppose that $C_1 \overset{X}{\approx} C_0$, $C_1 \not\subseteq \psi(X)$, and $C_0 \not\subseteq \psi(X)^{\mathrm{c}}$. Thus $C_1 \subseteq \psi(X)^{\mathrm{c}}$ and $C_0 \subseteq \psi(X)$, since ψ is connected. Pick $h \in C_1$, then $C_1 \cup C_0 \subseteq \gamma_h(X \cup \psi(X))$. However, $\gamma_h(X) \cup \gamma_h(\psi(X)) = C_1$ and therefore we get a contradiction with (9.2).

(b): Next, we assume that $\mathcal{C}$ is a strong connectivity class and that the operator ψ is stable; we show that (9.2) holds. If (9.2) does not hold, then there exist h and X for which $\gamma_h(X \cup \psi(X))$

is strictly larger than $\gamma_h(X) \cup \gamma_h(\psi(X))$. Obviously, $h \in X$ or $h \in \psi(X)$; defining $C = P(X, h)$ we have $C \subseteq \gamma_h(X \cup \psi(X))$ and $C \subseteq \gamma_h(X) \cup \gamma_h(\psi(X))$. There must exist another zone C' of $P(X)$ such that $C' \subseteq \gamma_h(X \cup \psi(X))$ but $C' \not\subseteq \gamma_h(X) \cup \gamma_h(\psi(X))$. From our assumption that $\mathcal{C}$ is a strong connectivity class, we conclude that there exists a path $C = C_1 \sim C_2 \sim \cdots \sim C_n = C'$ with C_k parts of $P(X)$ and $C_k \subseteq \gamma_h(X \cup \psi(X))$ (we also choose $C_k \neq C_{k+1}$). The zones C_k lie in X and X^{c}, alternatingly. Since $C_k \subseteq \gamma_h(X \cup \psi(X))$, the zones in X^{c} lie in $\psi(X)$. But then, by (9.1) the two neighbours C_{j-1}, C_{j+1} of such a zone C_j lie also in $\psi(X)$. However, this yields that all zones, including C', lie in $\psi(X)$, a contradiction. ∎

As a matter of fact, Crespo et al [10, 8] use relation (9.2) to define stable connected operators.

9.4. Proposition.
(a) Let ψ_i, $i \in I$, be connected stable operators, then $\bigvee_{i \in I} \psi_i$ and $\bigwedge_{i \in I} \psi_i$ are stable, too.
(b) If ψ_1, ψ_2 are connected stable operators, then $\psi_2\psi_1$ is stable.
(c) If ψ is connected and stable, then ψ^* is stable.

Rather than giving a formal proof (which is rather straightforward) we will sketch the intuition behind, say, (b) in Figure 9.2.

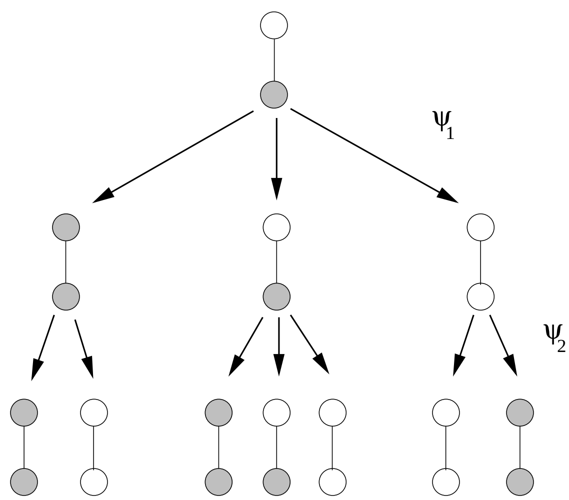


Fig. 9.2. The composition $\psi_2\psi_1$ is stable if both ψ_1 and ψ_2 are stable.

Two adjacent zones C_1, C_0 of X with opposite colours both receive the same colour or remain unchanged if a stable connected operator ψ is being applied to X. Thus, subsequent application of ψ_1 and ψ_2 leads to one of the configurations at the bottom of Figure 9.2: the colours at C_1 and C_0 cannot be changed both.

Our motivation for introducing stable operators is their usefulness in the investigation of connected filters and grain filters in the two forthcoming sections.

Recall that a (strong) *connected filter* is a (strong) filter that is a connected operator; see also Section 2.

9.5. Proposition. *Every strong connected filter is stable.*

Proof. Assume that ψ is a strong connected filter and that ψ is not stable. Then there is a set $X \subseteq E$ and C_1, C_0 with $C_1 \overset{X}{\approx} C_0$ such that $C_1 \not\subseteq \psi(X)$ and $C_0 \not\subseteq \psi(X)^{\mathrm{c}}$; refer to Figure 9.3 for a sketch in terms of the zonal graph of X and $\psi(X)$.

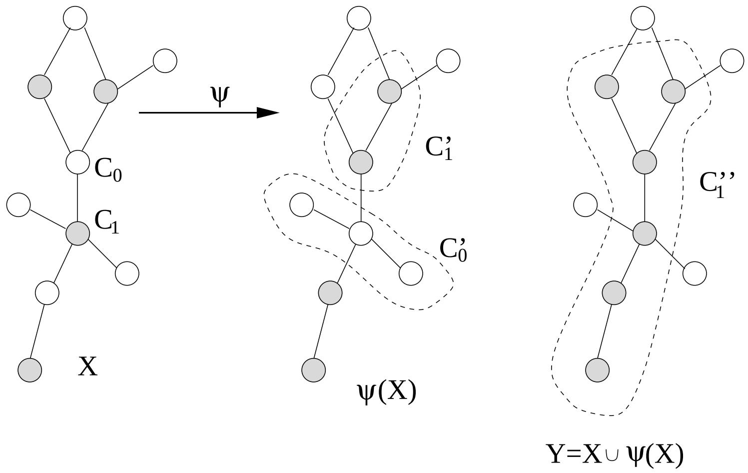


Fig. 9.3. See proof of Proposition 9.5.

Let C_1', C_0' be such that $C_1' \overset{\psi(X)}{\approx} C_0'$ and $C_0 \subseteq C_1'$, $C_1 \subseteq C_0'$. Define $Y = X \cup \psi(X)$. There exists a grain C_1'' of Y that contains $C_1 \cup C_1'$, since $C_1 \cup C_1'$ is connected. As $C_1' \Subset \psi(X) = \psi^2(X) \subseteq \psi(Y)$, we conclude that $C_1'' \subseteq \psi(Y)$. But, since ψ is strong, we have $\psi(Y) = \psi(X \cup \psi(X)) = \psi(X)$, whence we find that $C_1'' \subseteq \psi(X)$. In particular, we have $C_1 \subseteq \psi(X)$, a contradiction. ∎

The converse result is not true, however: there exist stable connected filters that are not strong. Consider the example in Figure 9.4 where E contains the vertices of a graph and $\mathcal{C}$ comprises all subsets of E that form a connected subgraph. Let the operator ψ on $\mathcal{P}(E)$ be defined as follows:

$$\psi(A) = B \text{ and } \psi(B) = B.$$

For other sets $X \subseteq E$, $\psi(X) = \varnothing$ if X contains not more than two points and $\psi(X) = E$ otherwise.

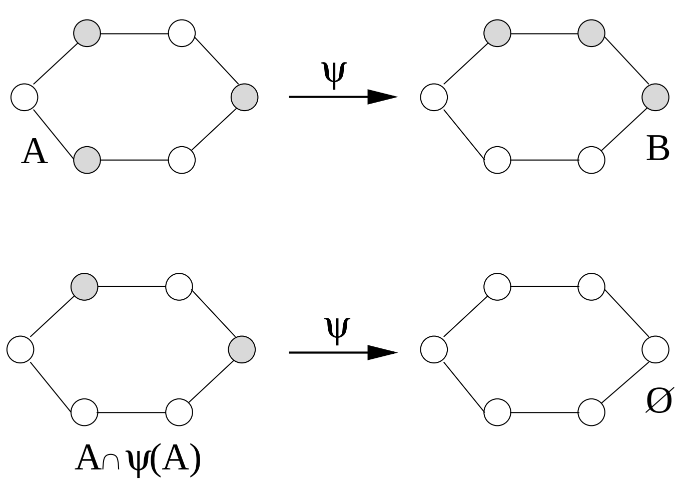


Fig. 9.4. The connected filter ψ is stable but not strong since $B = \psi(A) \neq \psi(A \cap \psi(A)) = \varnothing$.

The operator ψ is a stable connected filter, but it is not strong since $\psi(A) \neq \psi(A \cap \psi(A))$.

10. Grain filters

This section is concerned with grain filters. The main result (see Proposition 10.3) states that for an increasing grain operator, stability implies the strong filter property and vice versa.

Let us, by way of introduction, consider once more the special case $\mathcal{C} = \mathcal{C}_{\min}$. We have seen that in this case every operator on $\mathcal{P}(E)$ is connected and stable. In Section 8 we have learned that every grain operator is of the form

$$\psi(X) = (X \cap A) \cup (X^{c} \cap B),$$

where $A, B \subseteq E$; cf. (8.3). This operator is increasing if $B \subseteq A$. It is easy to verify (e.g, by using characteristic functions) that under this condition ψ is a strong filter. (In fact, one can easily show that ψ is increasing iff ψ is idempotent.)

Let us return to the general situation. In what follows, the following condition on the foreground and background criteria u, v plays an important role.

$$C_1 \approx C_0 \;\Rightarrow\; u(C_1) \vee v(C_0) = 1. \tag{10.1}$$

Obviously, if u or v is identically 1, then this condition holds trivially. Furthermore, as we show in our next result, this condition is somewhat stronger than (8.4); recall that the latter condition has been used to establish increasingness of the grain operator $\psi_{u,v}$.

10.1. Proposition. *Assume that $\mathcal{C}$ is a strong connectivity class. Let u, v be increasing grain criteria for which* (10.1) *holds, then* (8.4) *holds as well, i.e.*

$$u(\gamma_h(X \cup \{h\})) \vee v(\gamma_h(X^{c} \cup \{h\})) = 1,$$

if $X \subseteq E$ and $h \in E$.

PROOF. Let $X \subseteq E$ and $h \in E$. Without loss of generality we assume that $h \in X$. Putting $C = \gamma_h(X \cup \{h\}) = \gamma_h(X)$ and $D = \gamma_h(X^{c} \cup \{h\})$, we must show that $u(C) \vee v(D) = 1$. We distinguish two cases:

$D = \{h\}$: this means that every neighbour of h is an element of X, and hence of C. Define $Y = C \backslash \{h\}$, choose $k \neq h$ with $h \sim k$, then $k \in Y$. If $C' = \gamma_k(Y)$, then $C' \overset{Y}{\approx} \{h\}$ and from (10.1) we derive that $u(C') \vee v(\{h\}) = 1$. Since u is increasing and $C' \subseteq C$, this yields that $u(C) \vee v(D) = 1$.

D is larger than $\{h\}$: choose $k \in D$ such that $h \sim k$ and define $D' = \gamma_k(X^{c})$, then $C \overset{X}{\approx} D'$ and we conclude from (10.1) that $u(C) \vee v(D') = 1$. Since v is increasing and $D' \subseteq D$ this yields that $u(C) \vee v(D) = 1$. This concludes the proof. ∎

The following formal notation for the vertices of the zonal graph helps us to keep the proofs below compact and understandable.

10.2. Definition. Let $X \subseteq E$ be fixed. Denote by Π_X the family of subsets of vertices of the zonal graph of X that form a connected subgraph. In other words, an element $p \in \Pi_X$ corresponds with a collection $\{P(X,h) \mid h \in H\}$, where $H \subseteq E$, such that $\bigcup_{h \in H} P(X,h)$ is connected. Let $p, q \in \Pi_X$; we write $p \sim q$ if $P(X,h) \sim P(X,k)$ for some $P(X,h) \in p$ and $P(X,k) \in q$. If u is a grain criterion and $p \in \Pi_X$, $p = \{P(X,h) \mid h \in H\}$, then $u(p) := u(\bigcup_{h \in H} P(X,h))$. By $\Delta(p)$ we denote the element in Π_X comprising the zones in p and all of its neighbours:

$$\Delta(p) = p \cup \{P(X,k) \mid P(X,k) \sim P(X,h) \text{ for some } h \in H\}.$$

If $p \in \Pi_X$ comprises one zone of X, say $p = \{P(X,h)\}$, we write $p \doteq 1$ if this zone has colour 1 (i.e., $h \in X$) and $p \doteq 0$ if it has colour 0 (i.e., $h \in X^c$). Furthermore, if ψ is a connected operator, then $P(\psi(X),h)$ is a union of zones of X (including $P(X,h)$); we denote this collection by p_ψ. In mathematical terms:

$$p_\psi = \{P(X,k) \mid P(X,k) \subseteq P(\psi(X),h)\}.$$

We write $p_\psi \doteq 1$ if $h \in \psi(X)$ (meaning that all zones $P(X,k)$ in p_ψ have achieved colour 1) and $p_\psi \doteq 0$ if $h \notin \psi(X)$.

Observe that $p \subseteq p_\psi$ if $p = \{P(X,h)\}$ and ψ is a connected operator.

We are now ready to state the main result of this section.

10.3. Proposition. *Let u, v be grain criteria such that the grain operator $\psi_{u,v}$ is increasing and let $\alpha_u = \psi_{u,1}$ and $\beta_v = \psi_{1,v}$ be a grain opening and closing, respectively. The following are equivalent:*

(i) $C_1 \approx C_0 \;\Rightarrow\; u(C_1) \vee v(C_0) = 1$ *(i.e., condition* (10.1) *holds);*
(ii) $\psi_{u,v}$ *is stable;*
(iii) $\psi_{u,v}$ *is a strong filter;*
(iv) $\psi_{u,v} = \alpha_u\beta_v = \beta_v\alpha_u$.

Proof. Note first that u, v are increasing by Proposition 8.5. In what follows we delete the subindices u and v.

(i) $\iff$ *(ii)*: straightforward.

(i) $\Rightarrow$ *(iv)*: assume that (10.1) holds, we show that $\psi(X) = \alpha\beta(X)$ for every X. This amounts to showing that $p_\psi \doteq 1$ iff $p_{\alpha\beta} \doteq 1$ for $p = \{P(X,h)\} \in \Pi_X$. We distinguish two situations: $p \doteq 1$ and $p \doteq 0$.

$p \doteq 1$: If $p_\psi \doteq 1$, then $u(p) = 1$ and, since $p \subseteq p_\beta$ and u is increasing, we have $u(p_\beta) = 1$. This implies that $p_{\alpha\beta} \doteq 1$. If, on the other hand, $p_{\alpha\beta} \doteq 1$, then $p_\beta \doteq 1$ and $u(p_\beta) = 1$. If we would have $u(p) = 0$, then by (10.1), $v(q) = 1$ for $q \sim p$; this would imply $p_\beta = p$. But this is in contradiction with $u(p_\beta) = 1$. Thus $u(p) = 1$ which means that $p_\psi \doteq 1$.

$p \doteq 0$: If $p_\psi \doteq 1$, then $v(p) = 0$ and we get that $\Delta(p) \subset p_\beta$. From (10.1) we get that $u(q) = 1$ for $q \sim p$, hence $u(p_\beta) = 1$. Therefore $p_{\alpha\beta}$ has the same colour as p_β, that is $p_{\alpha\beta} \doteq 1$. If, on the other hand, $p_{\alpha\beta} \doteq 1$, then $p_\beta \doteq 1$ and $u(p_\beta) = 1$. This is possible only if $v(p) = 0$, and we conclude that $p_\psi \doteq 1$.

We have shown that $\psi_{u,v} = \alpha_u\beta_v$. Now, if (10.1) holds for u, v, then it holds for v, u as well, which means that $\psi_{v,u} = \alpha_v\beta_u$. Taking negations and using Proposition 8.3(a) we find that $(\psi_{v,u})^* = \alpha_v^*\beta_u^*$, i.e. $\psi_{u,v} = \beta_v\alpha_u$.

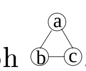
(iv) $\Rightarrow$ *(iii)*: It is a well-known fact [14] that $\beta\alpha$ is an inf-filter and that $\alpha\beta$ is a sup-filter. This implies that ψ is a strong filter.

(iii) $\Rightarrow$ *(ii)*: see Proposition 9.5. ∎

10.4. Corollary. *A supremum/infimum of strong grain filters is a strong grain filter.*

Proof. Combination of Proposition 8.3(b), Proposition 9.4, and the previous result. ∎

With some efforts, one can find a grain filter that is not strong. We present one particular example.

10.5. Example. Let $E = \{a, b, c\}$ contain the vertices of the graph . The edges of the graph induce a strong connectivity class on E. Define the grain criteria u and v as follows:

$$\begin{cases} u(C) = [a \in C] \\ v(C) = [b \in C \text{ or } c \in C] \end{cases}$$

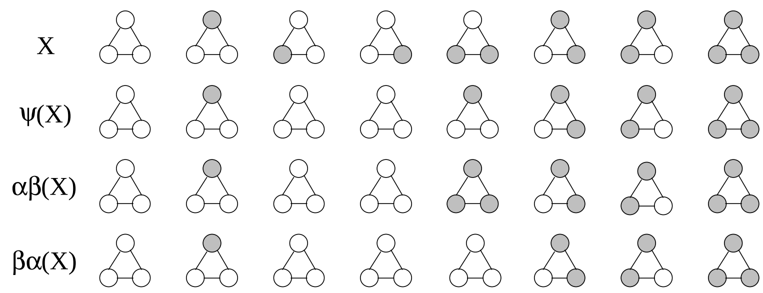


Fig. 10.1. ψ is a grain filter that is not strong. Furthermore, $\beta\alpha \leq \psi \leq \alpha\beta$.

Note that condition (10.1) does not hold: let $X = C_1 = \{b, c\}$ and $C_0 = \{a\}$, then $C_1 \overset{X}{\approx} C_0$, but $u(C_1) = v(C_0) = 0$.

In Figure 10.1 we compute $\psi_{u,v}$ as well as $\alpha_u\beta_v$ and $\beta_v\alpha_u$ for all possible subsets of E. It follows immediately that ψ is a filter; it is, however, not stable and therefore not strong.

In our next result we prove an interesting property of the invariance domain of a grain filter ψ (i.e., the family of sets X that satisfy $\psi(X) = X$).

10.6. Proposition. *Assume that E possesses a strong connectivity. Let ψ be a grain filter on $\mathcal{P}(E)$ with $\psi(X) = X$.*

(*a*) *If Y is a union of grains of X, then $\psi(Y) = Y$.*
(*b*) *If Y is a union of grains of X^c, then $\psi(Y^c) = Y^c$.*

PROOF. (a): First we prove that $\psi(Y) \subseteq Y$. Suppose not; since ψ is connected, $\psi(Y)\backslash Y$ consists of grains of Y^c. Let D be a grain of Y^c contained in $\psi(Y)\backslash Y$. We show that $D \cap X^c \neq \varnothing$. Suppose namely that $D \subseteq X$. The grain D must be adjacent to a grain C of Y, meaning that $C \cup D$ is connected. However, $C \cup D \subseteq X$, and we conclude that C cannot be a grain of X. But this contradicts our assumption that Y consists of grains of X. Thus $D \cap X^c \neq \varnothing$.

Since $D \subseteq \psi(Y)$ and ψ is increasing, also $D \subseteq \psi(X)$. This yields that $\psi(X) \cap X^c \neq \varnothing$, i.e., $X \cap X^c \neq \varnothing$, a contradiction. We conclude that $\psi(Y) \subseteq Y$, as asserted.

Next we show that $Y \subseteq \psi(Y)$. Take $h \in Y$, then $X(h) = Y(h) = 1$. Furthermore, $P(X, h) = P(Y, h) = \gamma_h(X)$. The fact that ψ is a grain operator in combination with Proposition 8.4 yields that $\psi(X)(h) = \psi(Y)(h)$. But $\psi(X) = X$ and we conclude that $\psi(Y)(h) = X(h) = 1$, that is, $h \in \psi(Y)$. This shows that $Y \subseteq \psi(Y)$.

(b): If ψ is a grain filter, then ψ^* is a grain filter too. Furthermore, $\psi^*(X^c) = X^c$. If Y is a union of grains of X^c, then $\psi^*(Y) = Y$, by (a). But this means $\psi(Y^c) = Y^c$. ∎

We illustrate this proposition by means of Figure 10.2. The figure at the utmost left shows a set X (along with its zonal graph) that is assumed to be invariant under a given grain filter. Our proposition gives us that the other sets depicted in this figure are invariant, too. The first three sets are built of grains of X. The arrows in the zonal graph indicate which grains are used as building blocks. The six sets at the right are built by using background grains, again indicated by arrows in the zonal graph.

In Proposition 7.6 we have shown that $\xi = \rho(\phi \mid \psi)$ is connected if ψ is connected. Furthermore, Proposition 7.7 says that $\check{\alpha} = \rho(\alpha \mid \mathsf{id})$ is a connected opening if α is an opening. Below we demonstrate how these results can be extended if ψ is a grain operator.

10.7. Proposition. *Let E possess a strong connectivity. Assume that ψ is a grain filter and ϕ an overfilter with $\phi \leq \psi$. Then $\xi = \rho(\phi \mid \psi)$ is a connected filter and $\psi\xi = \xi$. Dually, if ϕ is an underfilter with $\phi \geq \psi$, then $\eta = \rho^*(\phi \mid \psi)$ is a connected filter and $\psi\eta = \eta$.*

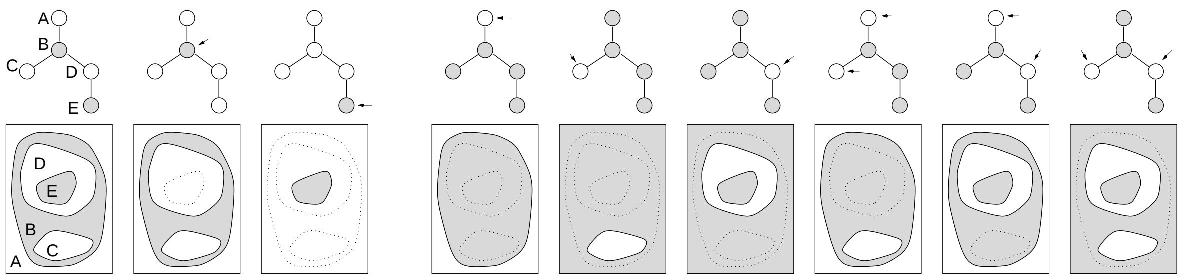


Fig. 10.2. The left figure shows a set invariant with respect to some grain filter ψ. Proposition 10.6 states that the other sets (in grey) shown in this figure are invariant, too.

PROOF. By the Duality Principle, we only need to prove the first result. We observe that $\xi(X)$ is a union of grains of $\psi(X)$ and, as $\psi(\psi(X)) = \psi(X)$, we deduce from Proposition 10.6 that $\psi(\xi(X)) = \xi(X)$. This yields that $\psi\xi = \xi$. Furthermore, $\phi \leq \psi$ implies that $\phi \leq \xi \leq \psi$. We get that $\xi^2 \leq \psi\xi = \xi$, and it remains to be proven that $\xi^2 \geq \xi$. We get

$$\begin{aligned}\xi^2 &= \rho(\phi\xi \mid \psi\xi) = \rho(\phi\xi \mid \xi)\\ &\geq \rho(\phi^2 \mid \xi) \geq \rho(\phi \mid \xi)\\ &= \rho(\phi \mid \rho(\phi \mid \psi)) = \rho(\phi \mid \psi) = \xi.\end{aligned}$$

Here we used that $\phi^2 \geq \phi$ and that $\rho(Y \mid \cdot)$ is idempotent (see $(R2)$ in Proposition 5.1). This concludes our proof. ∎

11. Alternating sequential filters

A basic method to construct morphological filters is by composition of openings and closings [14, 28]. Usually, one chooses monotonically decreasing sequences of openings ($\alpha_1 \geq \alpha_2 \geq \cdots$) and increasing sequences of closings ($\beta_1 \leq \beta_2 \leq \cdots$). Then

$$\begin{aligned}(\beta\alpha)_n &= \beta_n\alpha_n\beta_{n-1}\alpha_{n-1}\cdots\beta_1\alpha_1\\ (\alpha\beta)_n &= \alpha_n\beta_n\alpha_{n-1}\beta_{n-1}\cdots\alpha_1\beta_1\end{aligned}$$

are filters. Furthermore, these filters satisfy the *absorption laws*

$$\begin{aligned}(\beta\alpha)_n(\beta\alpha)_m &= (\beta\alpha)_n \geq (\beta\alpha)_m(\beta\alpha)_n,\ n \geq m\\ (\alpha\beta)_n(\alpha\beta)_m &= (\alpha\beta)_n \leq (\alpha\beta)_m(\alpha\beta)_n,\ n \geq m\end{aligned}$$

For grain openings $\alpha_n = \alpha_{u_n}$ we get monotonicity of the sequence α_n by taking a monotonically decreasing sequence u_n. Below we will show that we get some additional results for alternating sequential filters resulting from grain openings and closings. We start with the following general result.

11.1. Proposition. *Let E possess a strong connectivity. If $\psi_1, \psi_2, \ldots, \psi_n$ are strong grain filters, then the composition $\psi = \psi_n\psi_{n-1}\cdots\psi_1$ is a strong connected filter.*

PROOF. First we prove the following auxiliary result. Let $X \subseteq E$ and let C be a zone of the partition of $\psi(X)$. Suppose that $Y \subseteq E$ is such that

$$P(X,h) = P(Y,h) \text{ and } X(h) = Y(h),\ h \in C,$$

i.e., the zonal graphs of X and Y restricted to C coincide, then C is a zone of $\psi(Y)$ as well, and $\psi(X)$ and $\psi(Y)$ have the same value at C.

To prove this auxiliary result, we use that every ψ_k is stable. Consider the zones $P(X,h)$ of X, where $h \in C$; we distinguish:

internal zones: these are the zones that are not adjacent to some zone outside C;

boundary zones: these are zones that are adjacent to at least one zone outside C.

See Figure 11.1 for a visualisation of the zonal graph of X inside C.

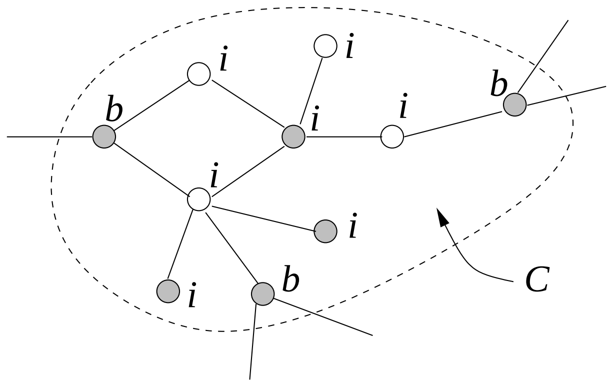


Fig. 11.1. Boundary zones are denoted by b, internal zones by i; see proof of Proposition 11.1.

We make the following important observation: the value of X at a boundary zone, as well as at its neighbouring zones outside C, does not change by application of any of the ψ_k. For, if some ψ_k would change the value at a boundary zone, it must also change the value at its neighbours outside C, since otherwise this boundary zone and the external zones would be merged. However, the stability of ψ_k does not allow that two neighbouring zones both change their value. Therefore, all the boundary zones inside C have the same value, namely the value of $\psi(X)$ at C. Now, if Y satisfies the condition above, its zonal graph inside C, as well as the classification into internal and boundary zones, is the same as for X (however, the same boundary zone may have a different number of external neighbours for X as for Y). To compute $\psi_n\psi_{n-1}\cdots\psi_1$ at zones of X inside C, information about zones outside C is not required. Therefore, $\psi(X)(h) = \psi(Y)(h)$ for $h \in C$.

We verify that ψ is an inf-overfilter, i.e., $\psi(\mathsf{id} \wedge \psi) \geq \psi$. Suppose that C is a zone of $\psi(X)$ with value 1. Let $Y = X \cap \psi(X)$, then Y satisfies the condition mentioned above, and we get that $\psi(X) = \psi(Y) = 1$ at C. Therefore, $\psi(Y) \geq \psi(X)$.

Dually, we can derive that ψ is a sup-underfilter. Thus ψ is a strong filter. ∎

11.2. Corollary. *Let E possess a strong connectivity and let u_k, v_k, $k = 1, 2, \ldots, n$, be increasing grain criteria and $\alpha_k = \alpha_{u_k}, \beta_k = \beta_{v_k}$, then $(\beta\alpha)_n$ and $(\alpha\beta)_n$ are strong filters.*

Note, however, that $(\beta\alpha)_n$ and $(\alpha\beta)_n$ are not grain filters in general. Note also that, in contrast to the classical case, Corollary 11.2 does not require that the sequences u_k and v_k are monotone. Nevertheless, it may be useful to impose this restriction in practical cases.

In the classical (i.e., non-connected) case, it is not possible to say which is larger, $\alpha\beta(X)$ or $\beta\alpha(X)$. When $\mathcal{C} = \mathcal{C}_{\min}$ and α, β are the grain opening and closing, respectively, given by $\alpha(X) = X \cap A$ and $\beta(X) = X \cup B$, one gets immediately that $\alpha\beta \leq \beta\alpha$, with equality iff $B \subseteq A$ (in which case $\alpha\beta = \beta\alpha$ is a strong grain filter; see Section 10). Somewhat surprisingly, the next result shows that the reverse inequality holds presumed that E possesses a strong connectivity.

11.3. Proposition. *Assume that E possesses a strong connectivity. Let α, β be a grain opening and closing on $\mathcal{P}(E)$, respectively, and $\alpha \not\equiv \varnothing$, $\beta \not\equiv E$. Then*

$$\beta\alpha \leq \alpha\beta.$$

In particular,

$$\alpha\beta\alpha = \beta\alpha \text{ and } \beta\alpha\beta = \alpha\beta.$$

PROOF. Recall the notation introduced in Definition 10.2, where $X \subseteq E$ is given. Let $p \in \Pi_X$ be a vertex representing a zone of X such that

$$p_{\beta\alpha} \doteq 1 \text{ and } p_{\alpha\beta} \doteq 0.$$

We must show that this leads towards a contradiction. Assume, furthermore, that

$$p \doteq 0;$$

the case $p \doteq 1$ is treated in a similar way. Since α is an opening, we get that

$$p_\alpha \doteq 0.$$

From the assumption that $p_{\beta\alpha} \doteq 1$, we find that $v(p_\alpha) = 0$. Since $p \subseteq p_\alpha$, this implies that $v(p) = 0$ as well. Therefore,

$$p_\beta \doteq 1.$$

Thus we have

$$p \doteq 0,\ p_\alpha \doteq 0,\ p_\beta \doteq 1,\ p_{\beta\alpha} \doteq 1,\ p_{\alpha\beta} \doteq 0. \tag{11.1}$$

From $p \doteq 0$ and $p_\beta \doteq 1$ we conclude that $\Delta(p) \subseteq p_\beta$. Thus the inclusions

$$\Delta^{2k}(p) \subseteq p_\alpha \text{ and } \Delta^{2k+1}(p) \subseteq p_\beta \tag{11.2}$$

have been established for $k = 0$ (where $\Delta^0(p) = p$). We use an induction argument to show that the relations in (11.2) are valid for every integer $k \geq 0$. Suppose that they hold for $k \leq m$. Let $q \in \Pi_X$ be a vertex contained in $\Delta^{2m+1}(p)$ but not in $\Delta^{2m}(p)$; then q must have the opposite value of p, i.e., $q \doteq 1$. Then $q \subseteq p_\beta$, and from the fact that $p_{\alpha\beta} \doteq 0$ we find that $u(q) \leq u(p_\beta) = 0$, that is, $u(q) = 0$. Therefore, $q \subseteq p_\alpha$. As $q \doteq 1$, every neighbour q' of q satisfies $q' \doteq 0$, hence $q' \subseteq p_\alpha$. This means that $\Delta(q) \subseteq p_\alpha$ for q contained in $\Delta^{2m+1}(p)$, i.e., $\Delta^{2m+2}(p) \subseteq p_\alpha$. We show that $\Delta^{2m+3}(p) \subseteq p_\beta$. Let $q \in \Pi_X$ be contained in $\Delta^{2m+2}(p)$ but not in $\Delta^{2m+1}(p)$, then $q \doteq 0$, hence $q \subseteq p_\alpha$. Since $p_{\beta\alpha} \doteq 1$, we find that $v(q) \leq v(p_\alpha) = 0$, and therefore $q \subseteq p_\beta$. Since $q \doteq 0$, every neighbour q' of q satisfies $q' \doteq 1$, hence $q' \subseteq p_\beta$. This gives that $\Delta(q) \subseteq p_\beta$ for $q \subseteq \Delta^{2m+2}(p) \setminus \Delta^{2m+1}(p)$. However, for $q \subseteq \Delta^{2m+1}(p)$ it is obvious that $\Delta(q) \subseteq p_\beta$, and we conclude that $\Delta(q) \subseteq p_\beta$ for every vertex $q \subseteq \Delta^{2m+2}(p)$. Thus, $\Delta^{2m+3}(p) \subseteq p_\beta$. This proves the assertion.

Now $\Delta^{2k}(p) \subseteq p_\alpha$ in combination with the assumption that E has a strong connectivity yields that p_α contains every vertex in Π_X; in other words $\alpha(X) = \varnothing$. Similarly we find that $\beta(X) = E$. However, we assumed explicitly that $\alpha \not\equiv \varnothing$ and $\beta \not\equiv E$. Therefore our starting assumption that there exists a p with $p_{\beta\alpha} \doteq 1$ and $p_{\alpha\beta} \doteq 0$ must be false, and the first result is proved.

To get, for example that $\alpha\beta\alpha = \beta\alpha$, we note that $\alpha\beta\alpha \leq \beta\alpha$ since $\alpha \leq \mathsf{id}$. On the other hand, using that $\alpha\beta \geq \beta\alpha$, we find $\alpha\beta\alpha \geq \beta\alpha\alpha = \beta\alpha$. ∎

We say that the sequence ψ_n has the *strong absorption property* if

$$\psi_n\psi_m = \psi_m\psi_n = \psi_n,\ n \geq m.$$

We prove the following extension of Corollary 11.2.

11.4. Corollary. *Assume that E is endowed with a strong connectivity. Let $u_1 \geq u_2 \geq \cdots \geq u_N$ and $v_1 \geq v_2 \geq \cdots \geq v_N$ be increasing grain criteria and $\alpha_k = \alpha_{u_k}, \beta_k = \beta_{v_k}$, then the sequences of strong filters $(\beta\alpha)_n$ and $(\alpha\beta)_n$ have the strong absorption property. Furthermore,*

$$(\beta\alpha)_n \leq (\alpha\beta)_n.$$

PROOF. For the strong absorption property we have to show only that $(\beta\alpha)_m(\beta\alpha)_n \geq (\beta\alpha)_n$ as the other inequality is satisfied even in the non-connected case; see above. Now

$$\begin{aligned}(\beta\alpha)_m(\beta\alpha)_n &\geq (\alpha)_m(\beta\alpha)_n \\ &= \alpha_m\beta_n\alpha_n(\beta\alpha)_{n-1} \\ &\geq \alpha_n\beta_n\alpha_n(\beta\alpha)_{n-1} \\ &= \beta_n\alpha_n(\beta\alpha)_{n-1} = (\beta\alpha)_n.\end{aligned}$$

The proof for $(\alpha\beta)_n$ follows by duality.

The inequality $(\beta\alpha)_n \leq (\alpha\beta)_n$ is a straightforward consequence of the previous result. ∎

11.5. Example. In both examples considered below, we use the same criteria for foreground and background grains. Therefore, the resulting filters are self-dual. Consider the space $E = \mathbb{Z}^2$ endowed with 8-connectivity.

(a) Consider the area criterion

$$u_S(C) = [\text{area}(C) \geq S],$$

where S is a nonnegative integer; see Example 8.1. In Figure 11.2 we illustrate the filters $(\beta\alpha)_n$ for $n = 1, 2, 3$, where $u_n(C) = [\text{area}(C) \geq S_n]$ and $S_1 = 5$, $S_2 = 20$, $S_3 = 100$. The noise-cleaning effect of these filters inside homogeneous regions is quite good; however, noise pixels adjacent to edges are not affected by these filters (as we have seen, this is a general property of connected operators).

We make the following observation with regard to the filters $\omega_{S,T} = \beta_{u_T}\alpha_{u_S}$. It is not difficult to verify that condition (10.1) holds for the pair $u = u_S$, $v = u_T$ if $S, T \leq 8$. Thus, by Proposition 10.1 and Proposition 10.3 we conclude that

$$\omega_{S,T} = \psi_{u_S,u_T} = \beta_{u_T}\alpha_{u_S} = \alpha_{u_S}\beta_{u_T}$$

is a strong grain filter if $S, T \leq 8$. If $S = T$, then $\omega_{S,T}$ is self-dual; see also [16].

(b) A second class of connected alternating sequential filters is obtained by using openings and closings by reconstruction; see Example 8.2. Consider the grain criterion

$$u_n(C) = [C \ominus B_n \neq \varnothing],$$

where B_n is a connected structuring element. In Figure 11.2, second row, we illustrate $(\beta\alpha)_n$ for $n = 1, 2, 3$, where B_1, B_2, B_3 are squares of size 3×3, 7×7, and 21×21, respectively.

Before we conclude this section, we present a short discussion about some related concepts in the literature. Serra and Salembier [30] call a filter ψ on $\mathcal{P}(E)$ a *ci-filter* ("connected invariant" filter) if the grains of $\psi(X)$ are invariant under ψ, i.e,

$$\psi\gamma_x\psi = \gamma_x\psi, \; x \in E.$$

A connected ci-filter is called a *filter by reconstruction.* Proposition 10.6(a) gives that a grain filter is a filter by reconstruction. It is easy to verify that every opening by reconstruction is

Fig. 11.2. Top, left to right: original image X (appr. 20% noise), and the area open-close filtered images $(\beta\alpha)_n(X)$ for $n = 1, 2, 3$; see Example 11.5(a). Bottom, left to right: reconstructed open-close filtered images $(\beta\alpha)_n(X)$ for $n = 1, 2, 3$; see Example 11.5(b).

a grain opening. For closings, however, this is not true. Consider the space $E = \{-1, 0, 1\}$ where $-1 \sim 0$ and $0 \sim 1$. Define β as follows: $\beta(\varnothing) = \varnothing$, $\beta(\{0\}) = \{0\}$, and $\beta(X) = E$ for all other sets. One verifies easily that β is a closing by reconstruction but not a grain closing. This observation shows in particular that being a filter by reconstruction is not a self-dual property: the fact that ψ is a filter by reconstruction does not imply that ψ^* is such as well. Crespo *et al* [11] define a closing by reconstruction as the dual of an opening by reconstruction, and a filter by reconstruction as a composition of openings and closings by reconstruction that is idempotent; see also [8].

12. Translation invariance

In classical morphology, translation invariance is an important issue. Here we shall briefly explain under which assumptions one may construct connected operators that are translation invariant; see also [7]. Most of our results are rather straightforward, and in these cases proofs will be omitted.

Throughout this section we assume that there exists a commutative group operation $+$ on E that we shall call 'addition'; for a systematic treatment of translation invariance in mathematical morphology the reader may refer to [17] or [14].

12.1. Definition.
(a) A connectivity class $\mathcal{C} \subseteq \mathcal{P}(E)$ is called *translation invariant* if $C \in \mathcal{C}$ implies $C_h \in \mathcal{C}$, for every $h \in E$.
(b) An adjacency relation $\sim$ on $E \times E$ is *translation invariant* if $x \sim y$ implies $x + h \sim y + h$ for $x, y, h \in E$.

Obviously, if $\sim$ is translation invariant, then the associated connectivity class $\mathcal{C}_\sim$ is translation invariant as well.

12.2. Proposition. *Let $\mathcal{C}$ be a translation invariant connectivity class on $\mathcal{P}(E)$; the following relations hold:*

$$\gamma_{x+h}(X_h) = [\gamma_x(X)]_h$$
$$\rho(Y_h \mid X_h) = [\rho(Y \mid X)]_h$$
$$P(X_h, x+h) = [P(X,x)]_h$$

for $x, h \in E$ and $X, Y \subseteq E$.

Apart from a translation of the zones, the zonal graphs of X and X_h are identical if the underlying connectivity class is translation invariant; this is due to the fact that $C_h \sim C'_h$ if $C \sim C'$, for every $h \in E$.

A grain criterion u is said to be *translation invariant* if $u(C_h) = u(C)$ for $C \in \mathcal{C}$ and $h \in E$.

12.3. Proposition. *A grain operator $\psi_{u,v}$ is translation invariant iff both u and v are translation invariant.*

To conclude this section, we answer the question under which conditions the opening by reconstruction is a grain opening; see Example 8.2 for a special case.

12.4. Proposition. *Let $B \subseteq E$ be an arbitrary structuring element and consider the grain criterion $u(C) = [C \ominus B \neq \varnothing]$. Furthermore, let $\breve{\alpha}$ be the opening by reconstruction defined by $\breve{\alpha}(X) = \rho(X \circ B \mid X)$; then $\alpha_u \leq \breve{\alpha}$, with equality if B is connected. If we assume in addition that B is finite, then $\alpha_u = \breve{\alpha}$ (in particular, $\breve{\alpha}$ is a grain operator) if and only if B is connected.*

PROOF. To prove that $\alpha_u \leq \breve{\alpha}$ we need to show (see [14]) that $\mathrm{Inv}(\alpha_u) \subseteq \mathrm{Inv}(\breve{\alpha})$. Assume that $\alpha_u(X) = X$ and let C be a grain of X, then $u(C) = 1$, i.e., $C \ominus B \neq \varnothing$. However, this yields that $C \subseteq \breve{\alpha}(X)$. We conclude that $\breve{\alpha}(X) = X$. Assume that B is connected; we show that $\breve{\alpha} \leq \alpha_u$, that is $\mathrm{Inv}(\breve{\alpha}) \subseteq \mathrm{Inv}(\alpha_u)$. Let $\breve{\alpha}(X) = X$ and C a grain of X. Since B is connected, it must hold that $C \circ B \neq \varnothing$, i.e., $C \ominus B \neq \varnothing$. Thus $u(C) = 1$ and we conclude that $\alpha_u(X) = X$.

Finally, assume that B is finite; we show that B has to be connected if $\breve{\alpha} = \alpha_u$. Suppose not; then $\breve{\alpha}(B) = B$ but $\alpha_u(B) = \varnothing$ since B does not fit inside any of its grains. ∎

We were not able to prove this last result without the finiteness condition; note that $C \ominus B \neq \varnothing$ may hold for a grain C of B in case that B is infinite.

13. Final remarks

As we observed in the introductory section, connected morphological operators are different from classical operators in at least two respects:

(i) they require the introduction of a connectivity class;
(ii) their operation is governed by criteria on the level of the zonal graph rather than by structuring elements (on the level of individual pixels).

A special case of the criteria referred to in (ii) are the grain criteria which lead to grain operators. In the most general case, criteria on the zonal graph level can be rather complex. Consider, for example, the opening by reconstruction $\breve{\alpha}(X) = \rho(X \circ B \mid X)$, where B consists of two non-adjacent points. As we observed in Example 8.2, this opening is not a grain opening (but obviously, it is connected). In terms of the zonal graph, this opening is given by

$$\breve{\alpha}(X) = \bigcup \{C \cup C' \mid C, C' \Subset X \text{ and } C_h \cap C' \neq \varnothing\}$$

where h is the vector connecting the two points in B.

If we consider criteria that are local in the sense that they can be evaluated grain by grain (in particular, without knowledge about the underlying graph structure), then the resulting connected operator is a grain operator. As we have shown, such operators satisfy a number of interesting properties.

Various of the results stated in this paper can be extended to the grey-scale case. It is well-known that increasing set operators can be extended to grey-scale images using level sets [13, 14]. But the grey-scale case also poses several new theoretical challenges:

- criteria can include grey-scale and contrast information
- there are several possible extensions of the definition of a grain operator
- it is tempting to develop connectivities for functions that use also grey-scale information (some first steps in this direction have been made by Serra in [29], where he considers connectivities on complete lattices).

We will pursue such and other ideas in our future work.

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